Teaching guide

tsm 1

Book 1

special edition

CURRICULUM

QA
11
S46
bk.1
tch.gd.

CURR HIST

Seeing through mathematics, Book 1

Van Engen Harting Trimble Berger Cleveland Emerson

Ex libris
universitatis
albertaensis

Digitized by the Internet Archive
in 2017 with funding from
University of Alberta Libraries

https://archive.org/details/seeingthroughmat01vane_1

Teaching guide tsm 1 Book

special edition

Seeing through mathematics, Book 1

Van Engen Hartung Trimble Berger Cleveland Emerson

W. J. Gage Limited

Toronto

UNIVERSITY
OF ALBERTA LIBRARY

Responses to "Special challenge" exercises are given in this *Teaching Guide*. Answers for all other exercises and tests in *Seeing Through Mathematics, Book 1* are given in the *Response Verification Books for Seeing Through Mathematics Book 1*.

The authors of this book are:
Henry Van Engen, Professor of Education and Mathematics, University of Wisconsin;
Maurice L. Hartung, Professor of Education, University of Chicago;
Harold C. Trimble, Professor of Education, Ohio State University;
Emil J. Berger, Coördinator of Mathematics, St. Paul Public Schools, St. Paul, Minnesota;
Ray W. Cleveland, Supervisor of Elementary and Secondary Mathematics, Rahway Public Schools, Rahway, New Jersey;
A. B. Evenson, Assistant Superintendent, Secondary Education, Edmonton Public Schools, Edmonton, Alberta.

The following persons served as consultants in preparing this book:
E. Glenadine Gibb, Professor of Mathematics, State College of Iowa;
Warren A. Parnell, Mathematics Instructor, Northeast Intermediate School, Midland, Michigan;
Wayne H. Peterson, Head of the Department of Mathematics, Marcus Whitman Junior High School, Seattle, Washington;
James E. Stochl, Instructor, College of Education, University of Minnesota;
Ray Walch, Director of Mathematics, K-12, Public Schools, Westport, Connecticut;
Willa Catherine Ware, Mathematics Teacher, Louisville Board of Education, Louisville, Kentucky.

This book was prepared by the editorial staff under the direction of George E. Russell, Directing Editor of The Basic Mathematics Program, and Clarence E. Olander, Associate Mathematics Editor.
This book was designed by William Nicoll and illustrated by Paul Hazelrigg.

Printed and bound in Canada.
All rights reserved.

444220

Contents

This book is your *Teaching Guide* for the student's book, *Seeing Through Mathematics, Book 1*. The *Teaching Guide* has been designed to provide assistance in using the student's book. The three major sections of this *Teaching Guide* are described below.

Introductory remarks pages 10-40

The first section of this *Teaching Guide* will acquaint you with the authors' point of view concerning a mathematics curriculum and will provide an overview of the *Seeing Through Mathematics* program for grades 7, 8, and 9. Also included are a description of the content and special features of *Book 1*, suggestions for adapting teaching methods and the content to meet individual differences, and a discussion of teaching procedures.

Point of view 10
Overview of the program 12
Overview of Book 1 17
Special features of Book 1 30
Adapting method to individual differences 34
Adjusting content to individual differences 37
Teaching procedures for Book 1 37

Lesson notes pages 42-229

The second section of this *Teaching Guide* will be most useful in preparing your daily lesson plans. Lesson notes are included for each of the 97 lessons, for each "Special challenge," and for certain "Applying mathematics" problems. To help you find the lesson notes for a particular lesson, the number of the lesson and the number of the page on which it appears in the student's book are given at the top of each right-hand page in the *Teaching Guide*. The lesson notes are organized under these headings: Objectives, Content overview, Instruction notes, and Provision for individual differences.

Supplementary material pages 232-266

Material of a more general nature is contained in the third section of this *Teaching Guide*. This section includes eight supplementary tests, one for each of the seven units in *Book 1* and an end-of-book test, which you may reproduce and use to help evaluate the progress of your students. The following reference materials are included in this section: a listing of mathematical properties taught in *Book 1:* a bibliography for teachers; a bibliography for students; responses for the "Special challenge" exercises; and an index.

Supplementary tests (end-of-unit and end-of-book) 232
Mathematical Properties taught in *Book 1* 248
Bibliography for teachers 252
Bibliography for students 253
Responses to Special Challenges 256
Index 261

Introductory remarks

This part of the *Teaching Guide* includes the following items: Statement of the authors' point of view on mathematics for this area of the curriculum. Summary and explanation of the organization of the content of *Seeing Through Mathematics*, *Books 1*, *2*, and *3*. Detailed description of the content and features of *Book 1*. Plan for adapting the program to differing ability groups. Plan for using *Book 1* with differing ability groups.

Point of view

■ The authors of the W. J. Gage Basic Mathematics Program make these two basic assumptions: (1) In the next decade, teachers and parents will demand that the schools provide better instruction in mathematics to an ever increasing portion of the student population; (2) good mathematics is easier to teach and easier to learn than bad mathematics.

These assumptions make it necessary to answer the following questions: "*Why* teach more mathematics?" and "What is 'good' mathematics as opposed to 'bad' mathematics?"

One obvious spot that lacks mathematical "know-how" appears in the area of administrative planning. Students of our society agree that the future will belong to those groups of people who plan better than their neighbors. A business organization that plans its operations efficiently and has the foresight to prepare for demands before they arise is more likely to succeed than other business organizations that plan haphazardly if at all. Likewise, a group of military men, a university faculty, or the administrative staff of a large school system must plan effectively in this competitive age. There is an increasing realization among administrators, whether businessmen, generals, university presidents, or school superintendents, that the methods of mathematics provide the best aids to planning yet devised by mankind. Whether the need is to build a rocket that will answer commands more accurately than previous rockets or to devise a system of inventory control so that the number of pairs of size $10\frac{1}{2}$ socks needed at a particular time will be on hand, the methods of mathematics will help to get the job done.

As civilization moves into an era of ever increasing industrial complexity, more and more persons recognize the need to teach more mathematics to more school children. They recognize the need for a greater number of creative mathematicians and for a greater number of people who can adapt existing mathematical models to special problems, for example, the best distribution of warehouse locations for efficient distribution of manufactured products. They recognize, too, that the citizens of a democratic society must become intelligent consumers of information stated in mathematical language. Consequently, the authors of the W. J. Gage Basic Mathematics Program believe that the schools must quickly prepare themselves to teach more mathematics to more students.

As we prepare, then, to accomplish this task, we must assume that "good" mathematics is more teachable than "bad" mathematics, and that "good" mathematics is the easier to learn. What, then, is "good" mathematics? What is "bad" mathematics? A brief historical note may help to point up the distinction.

Mathematics developed slowly over a long period of time. Ideas developed by one civilization contributed to the growth of ideas in later civilizations. For example, ideas on geometry from Euclid's lifetime (330?-275? B.C.) mingled with ideas on positive and negative numbers from Fibonacci's lifetime (1170-1248). As mathematics evolved, some mathematicians worked with great care to develop precisely worded definitions that eventually were incorporated into mathematical systems. This organization of ideas led, in time, to the development of new mathematical concepts. Mathematics developed in this way is what we mean by "good" mathematics. Some of the

old mathematics was certainly good mathematics, and, for this reason, Euclid's writings on geometry still interest students of mathematics, although many new ideas concerning geometry have evolved in the works of Bolyai, Lobachevski, Hilbert, and others. Moreover, even today, important new uses are sometimes found for some very old mathematical concepts. For example, much of what we know about astronomy depends on mathematical ideas originated by the early Babylonians. It is evident that "good" mathematics is not necessarily new mathematics.

Sometimes mathematics is taught as a series of isolated facts that are utilized as a series of rules with little or no regard for an understanding of the basic ideas involved. Thus, mathematical theorems appear in isolation one from another. That is, the concepts and theorems are not related to a unified whole. Mathematics developed in this way is what we mean by "bad" mathematics. Such mathematics is difficult to teach and, consequently, to learn, because it is not organized as a system.

If one thinks of mathematics as a collection of isolated bits of information, then it is natural to teach such mathematics by a "tell and do" method. But, as educators now know, the student who is subjected to such a method, learns, at best, a great many isolated bits of information, and such unrelated fragments are difficult to retain. A person who remembers most parts of a story but forgets one key step is in a weaker position than the person who remembers the idea of the story and sets out to redevelop the details for himself. Thus, "bad" mathematics is hard to learn and harder to remember.

In contrast, the student who learns "good" mathematics acquires information as it is related to a larger whole. The student organizes his ideas into a logical whole, in much the same way that a person working with a jigsaw puzzle fits the pieces together to make a complete picture. When mathematics is viewed as a system, the individual definitions, assumptions, and properties take their places as contributing parts of the whole system.

An example may help clarify what we mean. Consider the task of solving conditions for equality, or "equations." To an Arabian mathematician of 800 A.D., each equation required a specific method for its solution. Later, the Italian mathematicians of the Renaissance period learned to generalize their methods, which enabled them to solve rather large classes of equations. But they, too, found themselves unable to pursue their work beyond a certain stage. For example, men of great ingenuity failed to find a general solution for polynomial equations of degree 5. However, since 1500 A.D., progress in mathematics has taken two directions: (1) the methods of the calculus have provided solutions of any desired precision, and (2) modern algebra has shown ways to determine whether or not certain problems are capable of solution. For instance, the problem posed by the Italian mathematicians was not only difficult; it was impossible.

Thus, mathematicians of the past 150 years have learned different ways in which to ask the same questions. They have developed new number systems and new algebras. Consequently, they have learned to view problems in new ways, and they now enjoy more productive ways of finding solutions of problems.

However, the very perfection of modern mathematical thinking may impede learning, since the exposition of a logical system may be extremely dull for both teacher and student. By the time a system has been invented and perfected, it often is not interesting just to hear or to read an explanation of the system. It is much more fascinating to be involved in the development of the system. If the student is guided to *discover* the relations

for himself, to participate in the discovery of mathematical generalizations, he will enjoy the learning and will either remember the details or be able to work them out again when he needs them. This is the way we think "good" mathematics should be taught.

The emphasis on "good" mathematics has some important implications for a program of mathematics. If the pieces are really to fit into larger and larger wholes, mathematics must be taught and learned as a system. For this reason, mathematics must be carefully taught from the very beginning. As an example, consider the dependence of number ideas upon the basically simple idea of one-to-one correspondence. This concept can be studied by the very young child. This experience will help him later in discovering, under wise guidance, the concept of function, which is a central idea in the study of mathematics.

For the reasons just outlined, the team of authors and editorial workers began in about 1945 to produce a program that started with grade 1. Scott, Foresman and W. J. Gage are continuing to build a program that will carry boys and girls at least through grade 12. The program is consistent and cumulative, and it focuses on key mathematical ideas. The general plan of procedure is to guide students as they make their own discoveries of mathematical ideas.

The development of such a program has taken the best thinking of many people. We make no claim that ours is the only possible program. But we are confident that we have produced a good program of "good" mathematics and, hence, we have produced a program that is easier to teach because it is more meaningful to learn.

We feel sure that you, as a teacher, will give serious thought to your contribution to the program. As a teacher, you have many of the same problems that were faced by the authors. You also have many problems that the authors did not have to face. It is your responsibility to try to see that each day's work clarifies the student's mathematical thinking as he progresses from grade 1 through grade 12. Our confidence in teachers has made us bold enough to build the Basic Mathematics Program.

Overview of the program

■ In keeping with our philosophy of teaching mathematics in a meaningful way, we have used certain basic concepts throughout *STM*.* These basic concepts—the ideas of set, set operation, condition, variable, mathematical structure, relation, logic, and problem solving—serve as "strands" that provide a broad perspective of the field of mathematics. These strands permeate *STM*, strengthening the presentation of our program and enabling students to view mathematics as a unified whole.

Set ideas are introduced in *STA*,* but they are extensively used in *STM*. Explicit use of set ideas in *STM* weaves this strand into the fabric of our mathematics program.

The idea of *set* is a primitive notion. Since set is considered a basic notion, it is considered an undefined term. *Point* is another example. This does not mean that the ideas of set and point cannot be related to anything else in the experience of the student, but only that some such terms must remain undefined in every logical system. Remember that, in organizing mathematical ideas, there must be a starting point. Consequently, it is necessary to use certain undefined terms, such as *set* and *point*, to define other terms. For example, a set is described as "a definite collection of ob-

* Hereafter in this *Teaching Guide* we refer to the *Seeing Through Mathematics* program as "*STM*" and the *Seeing Through Arithmetic* program for grades 1 through 6 as "*STA*."

jects, such as the students in a class." If this were considered as a definition, we would be obliged either to define "collection" and "object" or to consider both as undefined terms. If we should attempt to define all such mathematical terms, the reasoning would either become circular (that is, we would reach a point at which our sequence of definitions would lead us back to the same term), or we would have to develop an infinite sequence of definitions. The latter would be impossible to achieve.

In *STM 1*,* the student is introduced to set through many examples familiar to him—a set of books, a set of people, a set of planets, and so on. Then he learns about sets of numbers and sets of points. Observe that sets are treated as mathematical objects that have certain properties.

Two set operations are discussed in *STM 1*. One of these operations results in what is called the *intersection* of two sets; the other results in the *union* of two sets. The operation that results in the intersection of two sets gives meaning to the logical connective "and." This connective, in turn, is used in finding solution sets of compound conditions and, hence, is important in the *STM* problem-solving approach. In *STM 2*, these ideas are extended to include a study of compound conditions involving the connectives "or" and "not." These connectives are also used in finding solution sets of compound conditions and therefore are important to the *STM* problem-solving story. Set operations are also used in presenting geometric ideas; for example, in developing definitions of geometric figures as sets of points. The ideas related to sets and set operations help to integrate algebra and geometry.

* Hereafter in this *Teaching Guide*, we refer to *Seeing Through Mathematics, Book 1* as *STM 1*, *Seeing Through Mathematics, Book 2* as *STM 2*, and *Seeing Through Mathematics, Book 3* as *STM 3*. We also refer to *Seeing Through Arithmetic 5* as *STA 5* and *Seeing Through Arithmetic 6* as *STA 6*.

Clarity of language is important in communicating such ideas as those just outlined. At this point, we urge you to think for a moment about the distinction that must be made in our language when we are discussing ideas, on the one hand, and, on the other hand, names for these ideas. A *number*, for example, is an idea—a mental construct. A *numeral*, however, is a physical thing; it can be written or spoken, and it names the number. The number-numeral distinction is now accepted by all programs. Similar language distinctions are desirable in the study of mathematics.

Throughout *STM*, careful attention is given to the distinction between the *symbols* used as a means of communication in mathematics and the *ideas* expressed by these symbols. This distinction between names and referents of the names is an integral part of the entire program.

The ideas of set, conditions, and variables help to give precise meaning to topics that have been presented mechanically in traditional texts. By means of these ideas and others, it is possible to unify the presentation of many types of conditions, such as conditions for equality, conditions for inequality, and conditions for equivalence.

Another use of these ideas is to give the student a powerful and sound approach to problem solving. After he is thoroughly familiar with conditions in one variable (as structuring certain problems), he studies conditions in two variables. These conditions provide motivation for introducing the ideas of an ordered pair and sets of ordered pairs. All this work involves familiarity with mathematical sentences, which is helpful in studying the properties of operations, since the use of sentences that employ symbols of mathematics permits us to write concise sentences instead of cumbersome sentences that use words. Still another use of the ideas of condition and variable is to unify geometric and algebraic ideas.

The variable in a geometric condition may be replaced by points from a universe, such as a line, plane, or space. The solution set, if it is not the empty set, is a geometric figure.

Geometry, which is developed in a sequential manner in *STM*, includes both nonmetric and metric geometry. Nonmetric geometry, introduced in the first unit of *STM 1*, is centered about the ideas of geometric point sets and is integrated throughout the student's work with sets and conditions. The concepts of union and intersection are applied to sets of points as well as to sets of numbers, and extensive use is made of the set idea in definitions of geometric objects. For example, an angle is defined as the union of two noncollinear rays that have a common endpoint.

After the introduction of the rational numbers of arithmetic in *STM 1*, metric geometry is studied in *STM 2*. The measure of a segment and the measure of an angle are introduced as number differences, which enables the students to observe metric properties concerning geometric figures.

In *STM 2*, geometry is studied from both a nonmetric and a metric point of view. The study of nonmetric geometry in a plane, which was begun in *STM 1*, is extended to nonmetric geometry in space. Ideas of metric geometry are also extended to include a study of 3-space. The use of graphs strengthens the integration of algebra and geometry. This "wedding" of point and number, one of the great unifying ideas in mathematics, enables students to view certain algebraic conditions from a geometric point of view and certain geometric conditions from an algebraic point of view.

In *STM 3* a further integration of algebra and geometry is accomplished through a study of vectors, including a study of the algebra and geometry of vectors in a plane.

As certain ideas occur and recur throughout mathematics, the student discovers that ideas give rise to basic structures, or patterns. One such structure is the concept of a *number system*, a central idea in the study of mathematics. Traditionally, in the teaching of arithmetic, the idea of a number system has been confused with that of a numeration system. This confusion came about, in part, because names and their referents have not been carefully distinguished. In *STM*, a number system is treated on the idea level and is carefully developed as a set of numbers, together with the operations of addition and multiplication, and the commutative, associative, and distributive properties of these operations. A numeration system, which is a method for naming numbers, is quite another thing.

The first number system developed in *STM 1* is the system of natural numbers. This system is developed by using the ideas of set and one-to-one correspondence and involves a consideration of the set of natural numbers and the operations of addition and multiplication as defined for the set of natural numbers. The student learns that the set of natural numbers is closed under both operations. He then concludes that the set of natural numbers, together with the operations defined, satisfies the requirements for a number system.

The authors chose to develop the basic ideas of a number system by using the set of natural numbers because it is easier for the student to grasp ideas in connection with these familiar numbers. Note, however, that the students not only work with infinite sets of numbers, such as the set of natural numbers, but they also observe that a finite set of numbers with two operations may form a number system. In this way, they broaden their understanding of the concept of a number system. We also provide a considerable amount of work in elementary number theory, a topic that challenges the students' intellect, provides a better understanding of some of the properties of the

natural numbers, and affords many excellent opportunities for making discoveries and generalizations.

The work on number systems, as continued in *STM 1*, might be entitled "Developing new numbers from old." In fact, much of the mathematics in grades 7, 8, and 9 is concerned with this purposeful extension of number systems. The properties developed for the natural numbers on the basis of the properties of sets and set operations are used in developing the system of rational numbers of arithmetic. The rational numbers of arithmetic are sometimes thought of as the "nonnegative rational numbers."

To develop the rational numbers of arithmetic, ordered pairs of natural numbers are introduced as fractions. The students then discover that certain fractions are equivalent and that there is an infinite set of fractions that are equivalent to each other. This infinite set of equivalent fractions is defined as a rational number of arithmetic. The *system* of rational numbers of arithmetic includes the set of rational numbers of arithmetic, two operations (addition and multiplication), and certain properties of these operations. The properties of operations of the rational numbers of arithmetic are shown to depend on the properties of operations of the natural numbers. Two other sets of numbers that are considered in *STM 1* are the non-zero natural numbers (called the counting numbers) and the set of non-zero rational numbers of arithmetic.

The system of rational numbers of arithmetic is an *extension* of the system of natural numbers. "Extension" may be described in the following way. The set of natural numbers is isomorphic to a subset of the rational numbers of arithmetic. That is, {0, 1, 2, . . .} and $\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \ldots\}$ behave in the same way under the operations of addition and multiplication. Therefore, the system of rational numbers of arithmetic may be considered an extension of the system of natural numbers. (See pages 342 and 343 of *STM 1*.)

In *STM 2* the rational-number system is developed as an extension of the system of rational numbers of arithmetic. Students learn that the rational-number system is made up of the positive rationals, the negative rationals, and zero, together with the operations of addition and multiplication and certain related properties. In *STM 2* the development of new number systems is continued with the rational-number system.

It is too difficult to use a definitional, constructive approach to develop the system of real numbers, and, hence, an axiomatic approach is used in developing this system. The real numbers are characterized as a set of elements that satisfy the postulates of an ordered field. This set contains all the rational and irrational numbers. Thus, the students have the real numbers, which are fundamental to algebra, available early in the eighth grade.

In *STM 3*, problems from the physical world are used extensively to motivate the study of vectors. This study contributes still more to the all-important idea of a mathematical system. In this study, the students have another example of a system. In this system, each element is a vector.

The idea of a relation as a set of ordered pairs is another recurring idea that is carefully and sequentially developed throughout *STM*. In *STM 1*, ordered pairs of numbers are used to interpret situations in the physical world. Sets of ordered pairs are then introduced as solution sets of conditions in two variables. These solution sets are represented by means of graphs. In *STM 2*, solution sets of conditions in two variables are used in the study of formulas with two variables. Variation of different types is introduced in *STM 2*, and certain similarities that exist among the various types are observed. In *STM 3* a relation is defined for the student as a set of ordered pairs;

that is, a set K is a relation if and only if each element of K is an ordered pair.

A special type of relation (a function) is studied with some care in *STM 3*. A function is defined as a relation in which no two distinct pairs have the same first component. An understanding of functions is necessary for the study either of mathematics or of any area that requires an extensive application of mathematics. For example, the study of acceleration in physics is dependent on an understanding of functions.

In the *STM* program, the concept of a mathematical system is constantly being developed. In particular, the rôle of logic in the development of a system is emphasized. The idea of proof, of course, involves logical deduction. Proof is introduced in an intuitive way in *STM 1* and *STM 2*, and, at this level, logic is used implicitly in developing ideas concerning operations on sets, number systems, and the properties of operations. The logical connective "and" is introduced in *STM 1*, the connectives "or" and "not" in *STM 2*. During the course of this work, the student learns how to develop new properties from properties that have already been accepted. He learns also that certain statements follow logically from accepted statements called "assumptions." Thus he discovers that certain properties can be derived by the use of these assumptions.

In *STM 2* the student studies deductive proof in both algebraic and geometric situations. Later, in *STM 3*, an entire unit is devoted to the study of logic and the nature of proof. The relationship between "and" and intersection of sets, "or" and union of sets, and "not" and complement of sets is thoroughly reviewed. The notion of proof is dealt with in detail, and the student uses both sets of statements and inference to arrive at logical conclusions. From this point on, many exercises in algebra, geometry, number theory, probability, and statistics take the form of a proof.

Although *STM* recognizes the importance of pure mathematics, which relates to the development of mathematical ideas in their own right, it must not be assumed that practical problem solving is neglected. Each unit in *STM* contains lessons on problem solving and on applied mathematics. Applied mathematics relates mathematics to situations in the physical world. Problem solving is included at all appropriate and desirable points. As new types of conditions are introduced, these conditions are applied to problem solving, and a wide variety of applications from science, geography, business, and other fields is used to illustrate the application of mathematical ideas. In this way, problem solving becomes a vital and integral strand in the entire *STM* program.

The problem-solving approach is simple, yet effective. The students are taught first to translate the structure of a problem situation into a mathematical condition. They then write a mathematical sentence that expresses a condition for the problem. They find the solution set of the condition and use this solution set to get an answer (or answers) for the problem. This problem-solving approach provides cohesion, because it brings together ideas on sets, conditions, variables, connectives, set operations, and solution sets. Problems that involve conditions for equality, inequality, and equivalence are used throughout *STM*, and as much attention is given to inequalities as to equalities. Problems involving inequalities lead to important application in grade 9, where linear programming problems that relate to systems of conditions are considered.

It is possible to unify and simplify problems concerning rates and comparisons, such as problems involving per cent and problems that in a traditional text are referred to as "ratio and proportion" problems. This unification and simplification are achieved in *STM 1* and *STM 2* within

the framework of proportional relations considered as sets of equivalent ordered pairs.

Remember that *STM* is *not* organized around "social" arithmetic. However, "social" arithmetic is included whenever it is appropriate in terms of the mathematical ideas being developed.

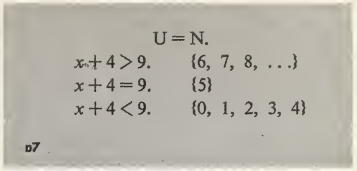
It should be clearly evident by now that there are two major themes in the *STM* program: (1) understanding of mathematical ideas (not memorization of number facts); and (2) integration of mathematical ideas (not compartmentalization).

Overview of Book 1

■ *STM 1* is the first book of the *STM* program in mathematics for grades 7, 8, and 9. This book emphasizes some of the big ideas of mathematics, but does not neglect computational skills. It is modern in spirit, but does not overlook traditional content. It not only reveals the logical structure of mathematics, but also shows the application of mathematics in many fields of human endeavor. *STM 1* blends theory with practice in such a way that students will be prepared for the study of either applied or pure mathematics.

There are seven units in *STM 1*, the first of which is entitled "Sets, conditions, and variables." The decision to begin with these ideas was partly motivated by the desire to have available, throughout the program, mathematical sentences to translate the structure of problem situations into the language of mathematics. The concepts and language of sets are convenient and appropriate aids in the study of mathematical sentences.

One such concept is that of replacing the variable in a condition by members of a universe to obtain a true statement. This concept is one of the most powerful ideas in a modern program of mathematics for grades 7, 8, and 9 because it not only unifies but also simplifies the study of conditions for equality, and thus provides a sound basis for problem solving.


U = N.

$x + 4 > 9$. {6, 7, 8, . . .}

$x + 4 = 9$. {5}

$x + 4 < 9$. {0, 1, 2, 3, 4}

07

Page 24. Universe set, condition, solution set of condition

The ideas of set and condition are also basic in the other seven units. These ideas give precise meaning to the concept of a relation. They are used when sets of points are considered in geometry. The concept of number is made more meaningful by the consideration of equivalent sets.

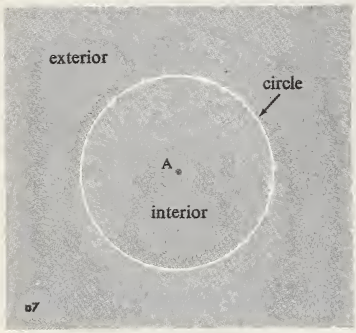
The idea of set is initiated by the study of open sentences (that is, sentences containing placeholders). In this connection, the name-referent distinction is important. The open sentence *expresses* a condition; the placeholder *expresses* a variable. Appropriate terminology, such as subset, empty set, solution set, universe, standard description, finite sets, and infinite sets, is introduced. Although at first glance the set approach may appear to be extremely abstract, it actually is an intuitive approach and is easy to associate with the experience of the student.

The display* shown above appears on page 24 of the student's book and illustrates the universe set, some conditions, and their solution sets. You can follow the sequence of content by observing the illustrations from the student's book shown on these pages, slightly reduced and without color.

Lessons on geometry also appear in unit 1. In these lessons, the ideas of point, line, space, and plane, as well as properties relating these ideas, are developed. Congruent segments are presented

* The displays and definitions shown in this section appear in *STM 1*.

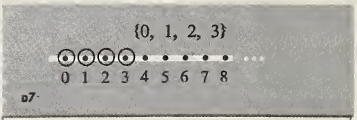
cir cle (sėr′kəl). The set of all points in a plane that satisfy $\overline{RX} \cong \overline{RT}$. The centre of the circle is point R. A segment one of whose endpoints is the centre of the circle and whose other endpoint is in the circle is a **radius** (rā′di əs) of the circle.



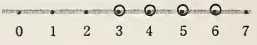
in te ri or of a circle (in tēr′ē ər). The set of all points in a plane that satisfy $\overline{AX} < \overline{AB}$. If $\overline{AB}$ is a radius of circle A, then $\{X \mid \overline{AX} < \overline{AB}\}$ is the interior of circle A. The universe for X is the plane that includes circle A. The points in the interior of a circle and in the circle are coplanar.

ex te ri or of a circle (eks tēr′ē ər). The set of all points in a plane that satisfy $\overline{AX} > \overline{AB}$. If $\overline{AB}$ is a radius of circle A, then $\{X \mid \overline{AX} > \overline{AB}\}$ is the exterior of circle A. The universe for X is the plane that includes circle A. The points in the exterior of a circle and in the circle are coplanar.

Pages 46, 48. The circle as the boundary of two regions



graph (graf). The graph of a set of numbers is a set of dots. Each dot in the graph represents a point associated with a member of the set of numbers. Each member of the set of numbers is associated with a point represented by a dot in the graph. The dots below form a graph of the set of natural numbers. The encircled dots form a graph of {3, 4, 5, 6}, a subset of the set of natural numbers.



Page 52. Natural numbers associated with points in a line

in terms of size and shape without the use of measure. The ideas of betweenness and separation are introduced and used to explain why one segment is "less than" or "greater than" another.

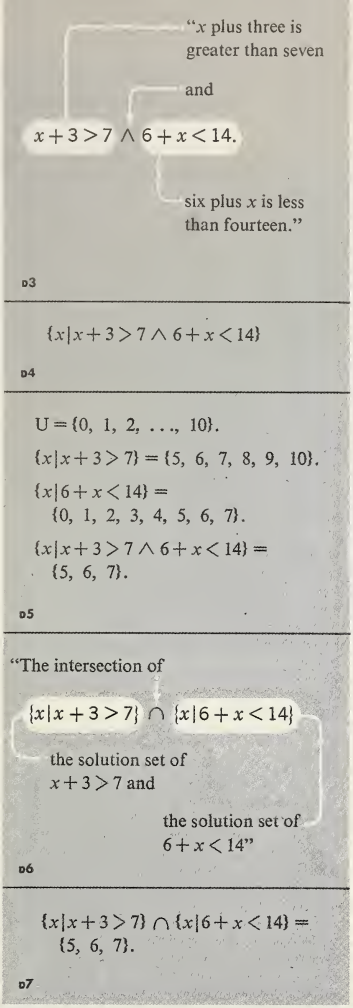
The first ideas concerning regions and boundaries are introduced in these geometry lessons. These ideas provide a "lead" to the study of the circle, its interior, and its exterior. The displays and definitions above show how the ideas of set and condition clarify the study of the circle.

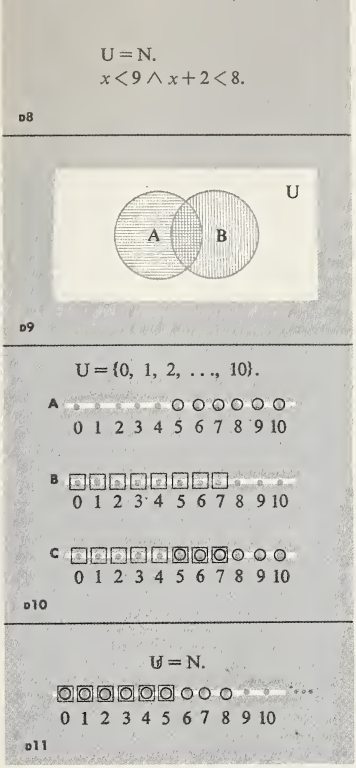
The natural-number line is also introduced in unit 1 and is used in connection with graphs of solution sets. This first association of numbers with points may be considered the informal introduction to analytical geometry that is integrated throughout the *STM* program.

Another less important but, nevertheless, significant reason for starting the year with sets, sentences, and geometry is that these topics are new and refreshing to most students. Too often students dislike seventh-grade mathematics because they do nothing but review the computational skills of arithmetic.

Computational review is not neglected, however. Such review is provided in "Checking up" exercises throughout the book and in "Learning computation" lessons 14 and 15 in unit 1. This review is included in unit 1 for another reason also: It is needed as preparation for solving problems that involve conditions of equality and inequality. Purposeful review like this is certainly more beneficial than drill and practice given in isolation.

Observe that the work in unit 1 involves only the natural numbers. In fact, the first six units in *STM 1* involve only the natural numbers or subsets of the natural numbers. This restriction was imposed because these units develop some very





Pages 91, 92. A compound condition and its solution set

important mathematical concepts, and it is desirable to develop these concepts within a simple framework. Thus, the notions of a condition, a variable, a set, an ordered pair, a relation, and a number system are first developed using only the natural numbers.

Later, the system of the natural numbers is extended to the system of the rational numbers of arithmetic (designated R_a). This development permits the student to concentrate on the important ideas he is learning, and it also emphasizes the importance of the universe.

In unit 2 the work on sets is extended to the operations that result in the intersection and the union of sets. Many non-mathematical examples and numerical examples are used to illustrate these operations. Venn diagrams are used as visual aids to extend the understanding of set relations.

Next, compound conditions involving "and" are studied, and their solution sets are related to the idea of intersection. Graphing is also used to illustrate the solution sets of such conditions.

Again attention is given to word problems that are related to social situations.

Mrs. Ames needed 10 hamburger buns for lunch. She did not have enough buns on hand. So she bought a package of 6. Then she had more than 12 buns. How many buns could she have had before she bought 6 more?

D1

A How do you know that Mrs. Ames had fewer than 10 hamburger buns to begin with? Use x as a variable for the number of buns she could have had before she bought more. Does the sentence at the left express one simple condition for the problem?

$x < 10$.

B Mrs. Ames bought a package of 6 buns and then had more than 12. Why do you again use x as a variable for the number of buns she could have had before she bought 6 more?

C Does the sentence at the left express another simple condition for the problem?

$x + 6 > 12$.

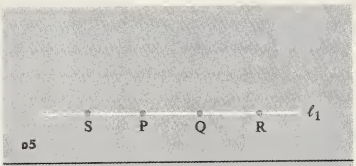
D Does the sentence at the left express the compound condition for the problem?

$x < 10 \wedge x + 6 > 12$.

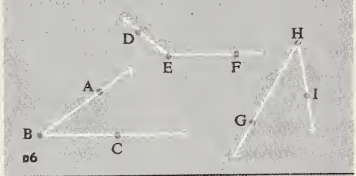
E How do you know that the set tabulated at the left is the solution set of $x + 6 > 12$?

$\{7, 8, 9, \ldots\}$

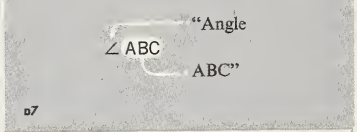
Pages 94, 95. The compound condition for a problem



D5

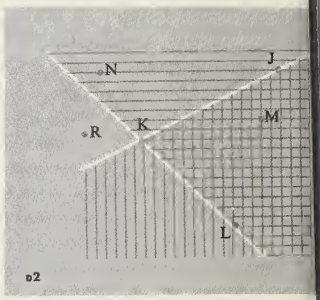


D6



D7

angle (ang′gəl). The union of two rays that have a common endpoint, but that do not form a line. The common endpoint is the **vertex** (vėr′teks) of the angle. Each of the rays is a **side** of the angle.



D2

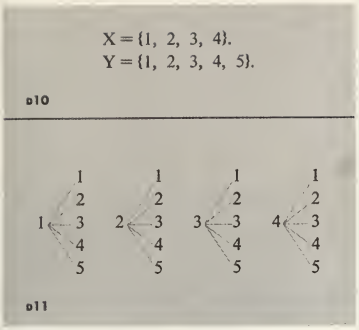
interior of an angle (in tēr′ē ər). The intersection of two half-planes. If $\angle ABC$ is any angle, then the interior of $\angle ABC$ is the intersection of the following two half-planes: the half-plane that is bounded by line AB and that contains point C and the half-plane that is bounded by line BC and that contains point A.

Pages 105, 114. Union and intersection of sets of points

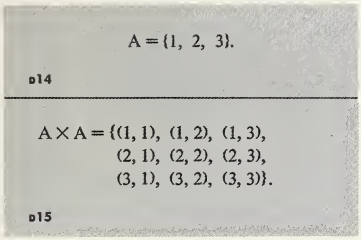
More geometry is included at the end of unit 2. The set operations introduced earlier in the unit can now be used to define many geometric figures. Thus, angles, triangles, polygons, parallel lines, parallel planes, and regions of the plane can all be defined in terms of intersections and unions of certain sets of points.

The concept of an ordered pair is introduced in unit 3, thus opening the door to new horizons of study by leading directly to a new kind of condition, namely, a condition in two variables. For such conditions, the solution sets are simple relations. Note, however, that the word *relation* is not used in the student's book and that the concept of a relation is not defined. The student merely works with solution sets of conditions. Later, in more advanced work, the student learns that a relation in N (the set of natural numbers) is a set of ordered pairs whose first and second components are members of N. In still more advanced work, he learns that relations can be determined by graphs, by tables, and by tabulation.

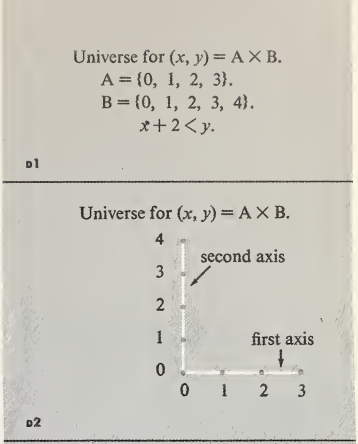
When ordered pairs have been introduced, the study of set is extended to include the notion of the Cartesian set of two sets. The student learns that a Cartesian set is the set of all ordered pairs that can be formed by matching each member of one set in turn with each member of a second set.

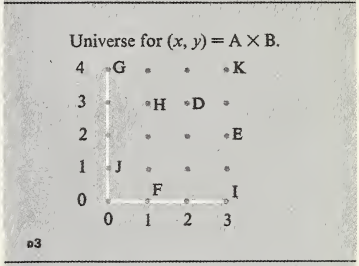


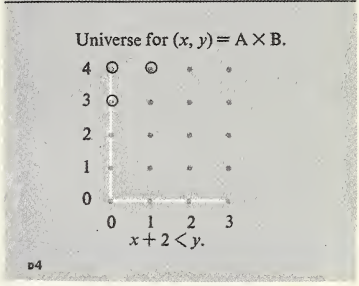
Page 128. Forming the Cartesian set of two given sets



Page 129. Tabulation of the Cartesian set of a set







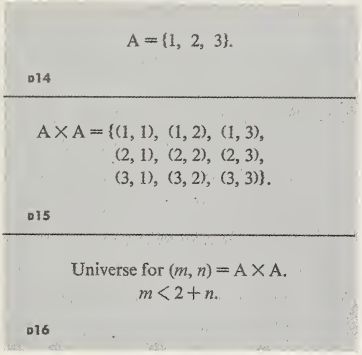
Page 131. Making graphs of sets of ordered pairs

He now can use Cartesian sets as universes for pairs of variables in conditions.

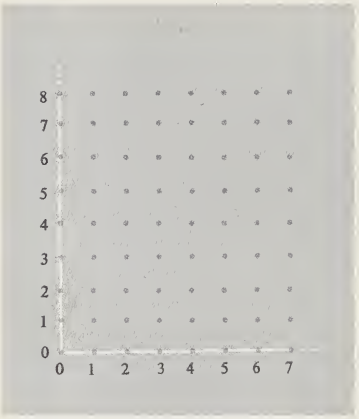
He also learns that it is possible to produce a Cartesian set from only one set of objects.

Now the work on graphs, which was heretofore related to the number line, can be extended to include graphs that represent points in a plane. In other words, the graphs take on a new dimension. This work on graphs should not be considered in isolation. Such graphs (of solution sets) provide insight into the nature of the condition whose solution sets are graphed. The graphs also are used to extend the ideas of analytical geometry, and they subsequently lead to the study of linear programming.

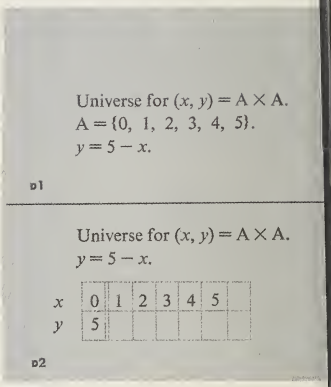
By this time, the student has learned that, for conditions with two variables, a set whose members are used to make replacements is needed for each variable. He has also learned how to find all the possible pairs he can use to make replacements when he is given such a set. In other words, he has learned how to form the Cartesian set that is the universe for a pair of variables. He also has learned how to find the solution set of a condition in two variables.



Page 129. A Cartesian set as the universe for variables



Page 140. A graph of the Cartesian set N×N



Page 143. Use of a chart to find solutions of a condition

As the student's next step, he learns about the Cartesian set N × N, which is an infinite set that contains all the ordered pairs of natural numbers. The display above, taken from page 140 of the student's book, is a graph of N × N.

As a further step, a chart is used to record replacements that yield solutions of a condition, and the chart is then used in making a graph of the solution set of the condition.

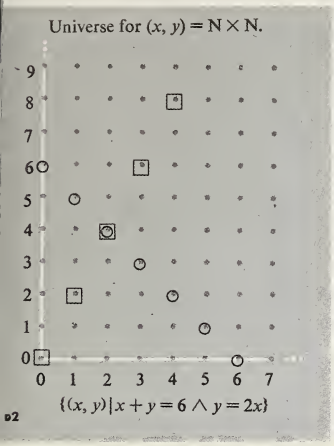
The student now is prepared to use N × N as a universe. The work that has just been completed on intersections, graphs, and solution sets containing ordered pairs, provides the necessary background for the study of compound conditions in two variables. The graphs of the two simple conditions contained in the compound condition reveal the solution set of the compound condition.

In unit 4 the concept of a relation is developed. However, as has been mentioned before, *relation* is not defined. Later, in more advanced work, a relation is defined as a set of ordered pairs. That is, a set R is a relation if each element of R is an ordered pair. This definition leads to a precise definition of function.

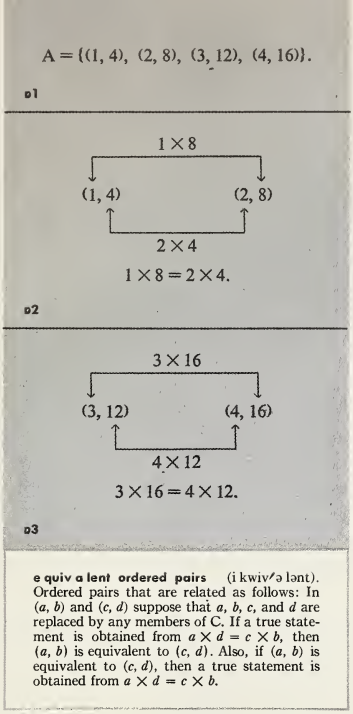
The work in unit 4 introduces proportional relations by first introducing equivalent ordered pairs.

The symbol used to express the idea of equivalence is introduced.

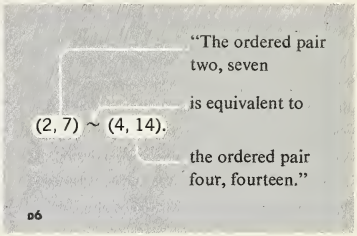
The student now learns that a proportional relation is a relation each of whose members is equivalent to each of the other members. The



Page 147.
A compound condition involving two variables


$A = \{(1, 4), (2, 8), (3, 12), (4, 16)\}$.
D1
1×8
(1, 4)
(2, 8)
2×4
$1 \times 8 = 2 \times 4$.
D2
3×16
(3, 12)
(4, 16)
4×12
$3 \times 16 = 4 \times 12$.
D3
e quiv a lent ordered pairs (i kwiv′ə lənt). Ordered pairs that are related as follows: In (a, b) and (c, d) suppose that a, b, c, and d are replaced by any members of C. If a true statement is obtained from $a \times d = c \times b$, then (a, b) is equivalent to (c, d). Also, if (a, b) is equivalent to (c, d), then a true statement is obtained from $a \times d = c \times b$.

Page 168.
A set of equivalent ordered pairs

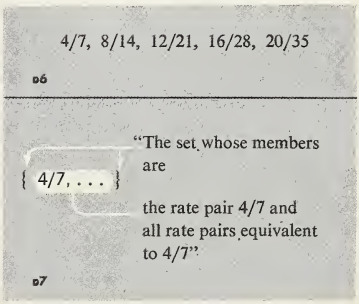


The symbol at the right below is used to express the idea of equivalence. D6 shows how to write and read a sentence that includes the symbol for equivalence. ~

Page 169. The symbol to express equivalent rate pairs

members of a proportional relation are called "rate pairs." He also learns that a set whose members are a given rate pair and all rate pairs equivalent to the given pair is an infinite proportional relation.

This "rate pair" terminology is introduced to give students the realization that they are working with ordered pairs that have a special application. The word *ratio* is introduced as a name of a rate pair. (In the W. J. Gage program a ratio is usually written with a diagonal bar to distinguish it from a fraction numeral.)



Page 174.
Tabulating an infinite proportional relation

Although different kinds of symbols can be used in expressing a rate pair, it is convenient to use the same kind of symbol all the time. In this book, we will agree to use a symbol like the one at the right in expressing a rate pair. 1/2

Page 173.
Use of diagonal to express a rate pair

Sixteen plants cost 8 dollars. At this rate, 4 of the plants cost how many dollars?

p1

U = C.
$16/8 \sim 4/x.$

p2

$16 \times x$
$16/8 \sim 4/x.$
4×8
$16x = 32.$

p3

U = C.
$8/16 \sim x/4.$
$32 = x(16).$
$32 = 16x.$

p4

Pages 176, 177.
Finding the missing component of a rate pair

This special notation for a ratio was adopted to emphasize the idea that a rate pair is a special kind of ordered pair. When a student sees a ratio, he should think of an ordered pair that is a member of a proportional relation and that it is used to solve rate or comparison problems.

STM 1 presents per cent problems within the general framework of comparison and defines per cent as the name of a special rate pair that represents a comparison and that always has a second component of 100. The usual three "cases" of per cent are easily handled within this framework.

It should be pointed out that much of the traditional work with per cents (discounts, interest, etc.) is covered in the applied mathematics of unit 4. The student is introduced to such applications of per cent through problems that are structured by conditions for equivalence.

It is impossible to overemphasize the careful problem-solving procedures used and developed in *STM 1*. A generalized method, based on recognizing a condition for a problem, is used throughout. Once a sentence that expresses the condition is written, it is comparatively easy for the student to determine the solution set of the condition.

The basic objective of unit 5 is to develop the properties of a modern numeration system. It was decided to include the work on numeration systems before any work was done on the system of natural numbers because the student is more familiar with the base-ten numeration system than he is with the rather subtle concept of a number system. The name-referent distinction, referred to

540 of the 720 students enrolled at Lee School live more than five blocks from school. What per cent of the students live more than five blocks from school?

a1

c What does the condition expressed below tell you about the rate pair whose first component you are to find? $540/720 \sim x/100.$

d How do you obtain $54000 = 720x$ from $540/720 \sim x/100$?

Page 190.
The condition for a per cent problem

2 In problem 1 you found what per cent the commission was of the total sales. This per cent of the total sales is the *rate of commission*. Mr. Monroe sold $5000 worth of merchandise. He received a commission of $450. What rate of commission did Mr. Monroe receive?

3 Mr. Martin's rate of commission is 4%. One month he sold $9650 worth of merchandise. What was the amount of his commission?

4 Miss Richardson gets a 5% commission on her total sales. One week her commission was $142. How many dollars' worth of merchandise had Miss Richardson sold that week?

5 Judy received a $5 discount on a dress that was regularly priced at $25. The discount was what per cent of the regular price?

6 In problem 5 you found what per cent the discount was of the regular price. This per cent of the regular price is the *rate of discount*. Mrs. Carey received a $45 discount on a television set that was regularly priced at $180. What was the rate of discount?

9 Mr. Harvey borrowed $500 and paid $30 interest at the end of one year. The amount of money on which interest is paid is the *principal*. The interest that Mr. Harvey paid was what per cent of the principal?

10 In problem 9 you found what per cent the interest was of the principal. This per cent of the principal is the *rate of interest*. Unless stated otherwise, the rate of interest is given for one year. Mr. Evans paid $10 interest on $200 that he had borrowed. What rate of interest did he pay?

Page 194.
Applications of per cent to business problems

several times before, is carefully preserved in this work with numeration systems. A secondary reason for including numeration systems here is the "newness" and consequent interest involved in the study of numeration systems with bases other than ten. The student brings an intrinsic interest into his work with these new ideas about numeration. Note the potential here for fostering the kind of creative thinking that is necessary in mathematical research.

There is no better way to achieve thorough understanding of the base-ten, or decimal, system of numeration than to undertake a careful study of various other systems that may be used to ex-

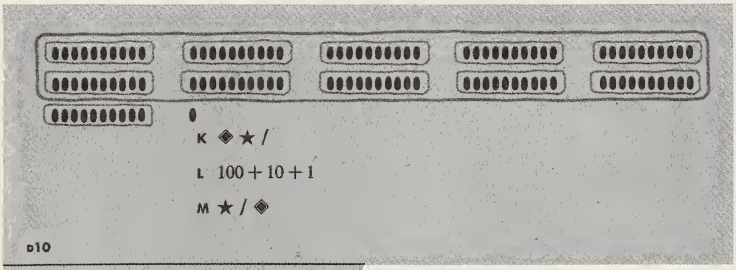
Certain rate pairs are equivalent to 6/8. The first component of each of these rate pairs is less than the second component. What are the rate pairs?

a5

$$h/k \sim 6/8 \wedge h < k.$$

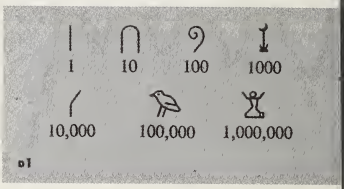
a6

Page 200.
The compound condition for an abstract problem



/ expresses the number 1.
★ expresses the number of objects equal to the base.
◆ expresses the number of objects equal to the base times the base.
● expresses the number of objects equal to the base times the base times the base.
For example, if the base is 3, then / = 1, ★ = 3, ◆ = 9, and ● = 27.
D11

Page 210.
A grouping system of numeration without place value



Page 219.
Symbols in the Egyptian numeration system

press numbers. In unit 5, tally, code, and grouping systems are studied and then used to develop the properties of a modern numeration system and to impress the students with the advantages of the base-ten system. The properties developed are the additive property, the property of repetition, the property of base, and the property of place value.

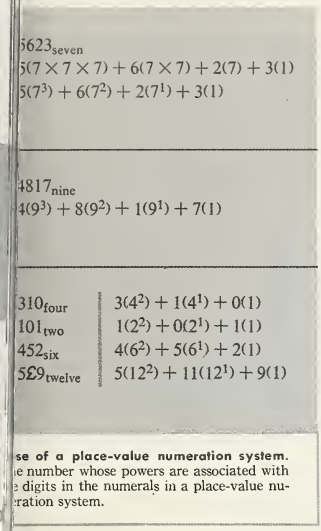
In this unit, the student works with the Egyptian, the Roman, and the Babylonian numeration systems. This work emphasizes the computational difficulties encountered in using these systems. The student learns that a motivating factor in the development of a modern system of numeration was to develop a system that would lend itself to convenient algorisms for computing.

Systems similar to the decimal system but involving different bases are next introduced. Study of these systems motivates the introduction of factors, exponents, and exponential notation. The final lesson in the unit extends the student's knowledge of the decimal system.

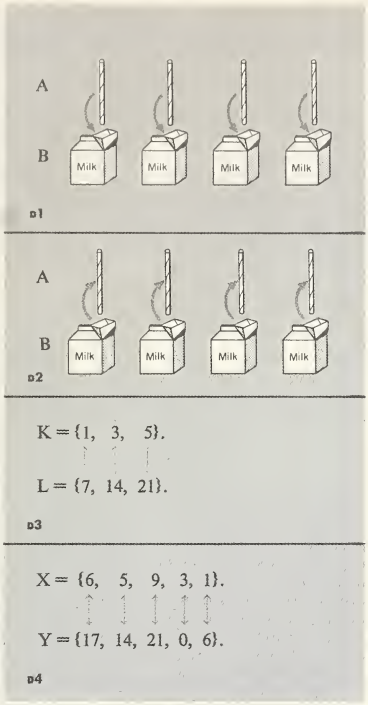
Unit 6 introduces a theme that recurs throughout the *STM* program. This theme, a number system, not only serves to unify, but also gives the student an example of a mathematical structure.

The student learns that a natural number is defined as a property of certain sets. These sets must be related to each other by a one-to-one correspondence. Mapping (the matching of members of one set with members of another set according to some specified plan) is considered first, because this concept leads to the idea of one-to-one correspondence. Various types of mappings can be used to explain other mathematical ideas.

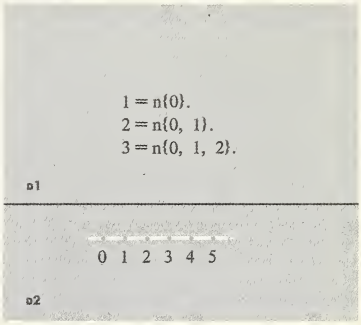
The student learns that each natural number is associated with a particular set, the standard set


5623_{seven}
$5(7 \times 7 \times 7) + 6(7 \times 7) + 2(7) + 3(1)$
$5(7^3) + 6(7^2) + 2(7^1) + 3(1)$
4817_{nine}
$4(9^3) + 8(9^2) + 1(9^1) + 7(1)$
310_{four} $3(4^2) + 1(4^1) + 0(1)$
101_{two} $1(2^2) + 0(2^1) + 1(1)$
452_{six} $4(6^2) + 5(6^1) + 2(1)$
$5\mathcal{E}9_{twelve}$ $5(12^2) + 11(12^1) + 9(1)$
se of a place-value numeration system.
e number whose powers are associated with
e digits in the numerals in a place-value nu-
eration system.

Page 230.
Converting numerals from one base to another



Page 246.
Mapping; one-to-one correspondence; equivalent sets



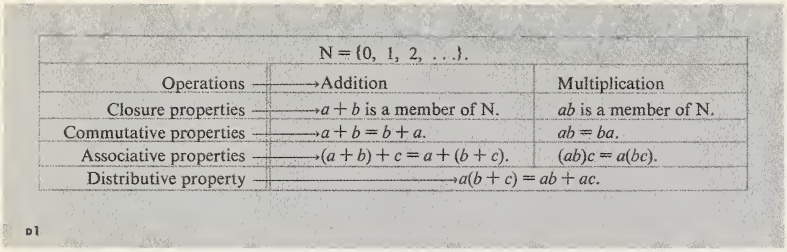
Page 253.
Natural numbers as properties of sets; ordering

of that number, and with all sets equivalent to the standard set. The empty set, { }, is the standard set that has no members. The natural number 0, associated with the empty set, tells how many members are in this standard set. When the natural number 0 has been defined, 0 can be used as a member of a standard set, {0}, that has one member, 0. The natural number 1 is defined as the number of members in {0}. The natural number 2 is defined as the number of members in the standard set {0, 1}. This process can be continued so that each natural number can be defined by using a standard set. Ordering of the natural numbers is accomplished by using the idea of proper subsets.

Since the natural numbers are associated with sets, it becomes necessary in *STM 1* to include zero with the set of natural numbers. This inclusion of zero in the set of natural numbers has become increasingly common in studies of number systems.*

Next the student learns that the natural numbers have the successor property and the property

* In *Introduction to Mathematical Philosophy*, Bertrand Russell includes zero as a natural number. In *Naive Set Theory* by Paul Halmos, in *Axiomatic Set Theory* by Patrick Suppes, and in *College Algebra* by A. Adrian Albert, zero is included in the set of natural numbers.

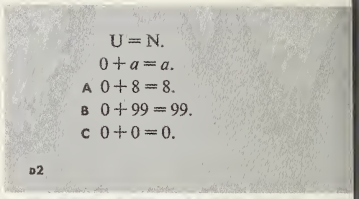


$N = \{0, 1, 2, \ldots\}$.

Operations	Addition	Multiplication
Closure properties	$a + b$ is a member of N.	ab is a member of N.
Commutative properties	$a + b = b + a$.	$ab = ba$.
Associative properties	$(a + b) + c = a + (b + c)$.	$(ab)c = a(bc)$.
Distributive property	$a(b + c) = ab + ac$.	

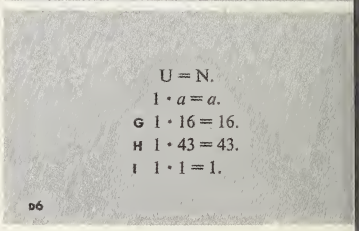
D1

Page 282.
Properties of the natural-number system



U = N.
$0 + a = a$.
A $0 + 8 = 8$.
B $0 + 99 = 99$.
C $0 + 0 = 0$.

D2



U = N.
$1 \cdot a = a$.
G $1 \cdot 16 = 16$.
H $1 \cdot 43 = 43$.
I $1 \cdot 1 = 1$.

D6

Pages 289, 291.
Identity elements for addition and multiplication

of betweenness. The system of natural numbers is obtained from a consideration of sets and from set operations and the properties of these operations. The operation of addition and the properties of addition (of natural numbers) are related to the union of disjoint sets. Multiplication and the properties of multiplication (of natural numbers) are related to the Cartesian set of two sets.

The properties of zero are derived from set operations using the empty set. The properties of the number one are derived from operations with a set containing one element.

The set of even numbers is shown to be a number system. Systems involving finite sets of numbers are also discussed. Two finite systems, the remainders of division by 5 and 3, are developed as number systems.

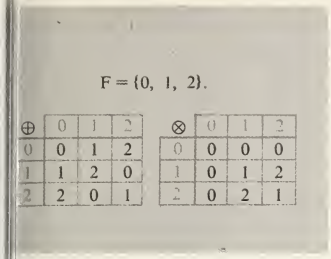
Number theory is another topic that enriches the study of other topics in mathematics. In unit 6, a prime number and a composite number are defined.

Here, too, the student studies the unique factorization property and uses the idea of prime factors to help him find the greatest common factor of two or more numbers.

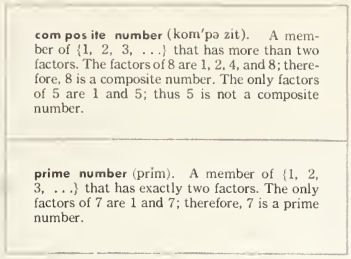
Customarily, a chapter dealing with fractions appears as the second or third chapter of a seventh-grade arithmetic textbook. Standardized tests on basic skills, sometimes given early in the school year, generally contain a number of items designed to measure proficiency in computation involving fractions expressed as common fraction numerals or decimal numerals.

In *STM 1*, however, this work with fractions is introduced in unit 7, and students using our program may be penalized if standardized tests on basic skills are given before the material in unit 7 has been presented in class. Obviously, in these cases, such test results are not valid. The testing should be delayed.

We delay the study of fractions in grade 7 because at this grade level we are primarily inter-

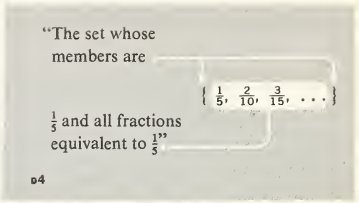


'age 288.
l number system
'hose set is
nite


com pos ite number (kom'pə zit). A member of {1, 2, 3, . . .} that has more than two factors. The factors of 8 are 1, 2, 4, and 8; therefore, 8 is a composite number. The only factors of 5 are 1 and 5; thus 5 is not a composite number.

prime number (prim). A member of {1, 2, 3, . . .} that has exactly two factors. The only factors of 7 are 1 and 7; therefore, 7 is a prime number.

Page 296.
Definitions used in the study of number theory



Page 314.
The set that is the rational number of arithmetic $\frac{1}{5}$

sted in the development of mathematical ideas uch as set, variable, condition, and rate pair. hese ideas are more easily understood by the tudent when he views them in a familiar setting. Hence, we elected to present these ideas within he framework of the natural numbers.

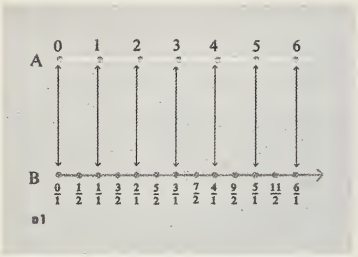
Once the basic ideas are developed, we develop he system of natural numbers. Again, a basic nathematical idea, a number system, is presented n a framework that is easy for the student to unerstand, the natural numbers. The system of atural numbers is then extended to the system of ational numbers of arithmetic. Then we begin he work with computation involving fractions xpressed as common fraction numerals and decinal numerals.

Traditionally, distinctions among such mathenatical ideas as fractions, rational numbers, rate pairs, and quotients has been haphazardly and nconsistently handled. In unit 7 of *STM 1*, these deas are carefully developed, and the distinctions nade are adhered to consistently.

A fraction is defined as an ordered pair that epresents a part of a whole. (You will recall that a rate pair expresses a rate or a comparison.) According to the definition in *STM 1*, a fraction does *not* mean "*a* divided by *b*." Instead, the student learns that a rational number is a set of equivalent fractions.

The student also learns that the fractions $\frac{1}{2}$ and $\frac{2}{4}$ are not equal. They are two different ordered pairs that represent two different part-whole situations. We therefore can say that the fractions are equivalent, that is, $\frac{1}{2} \sim \frac{2}{4}$. However, the rational number $\frac{1}{2}$ and the rational number $\frac{2}{4}$ are the same number, since either of these equivalent fractions indicates the same set. Consequently, when we are speaking of rational numbers, we can say $\frac{1}{2} = \frac{2}{4}$.

The ordering of the rational numbers of arithmetic is explained in two ways: first, by considering the position of the points in a number line and then by considering the products of the numerators and the denominators of the fractions that indicate two rational numbers. Multiplication and addition of rational numbers are then developed, as well as the properties for these operations of rational numbers. The rational



Page 342. Isomorphism between N and a subset of R_a

$U = R_a$.
$x + 0 = x$.

p4

$U = R_a$.
$x \cdot 1 = x$.

p9

Pages 361, 362. Identity elements for addition and multiplication

numbers are shown to have an important property that the natural numbers do not possess, namely, that it is possible to obtain the quotient of any pair of rational numbers, always provided that the divisor is not zero. The rationals have another property that the naturals lack. This is the density property. Between any two rational numbers there is always another rational number.

Lesson 78 in unit 7 discusses the isomorphism between the set of natural numbers and a subset of the rational numbers of arithmetic. The term "isomorphism" is not introduced to the student, but the idea is developed to help the student understand the use of mixed numerals in computation.

The identity-element properties for both multiplication and addition of rational numbers are presented in this unit, as well as the idea of a reciprocal of a rational number.

Nearly all the ideas just discussed are now used to develop the quotient property of rational numbers. Number theory appears again in the ideas of greatest common factor and least common multiple. The work on rational numbers and repeating decimals is also related to number theory.

Since the student now has a more versati number system to work with, more types of pro lems can be introduced in applied mathemati Such problems include the use of rate pairs wi rational number components, as well as per cen that involve rational numbers.

It is the hope of the authors and editors *STM* that this brief overview of *STM* content w whet your appetite and that you will recogni the great pedagogical strength of a *program* mathematics that is structured around fundame tal mathematical ideas. You will find, in *STM* that traditional concepts have not been discarde They are amplified and clarified by being fitte into a frame of reference. These concepts are longer presented as isolated, unrelated things do and to memorize. Instead, they form an orga ized body of knowledge that can be understoo through a way of thinking that has made o scientific age possible.

Special features of Book 1

■ *STM 1* presents a course in seventh-gra mathematics from a new (and different) point view. This new viewpoint, based upon a few fu damental ideas, provides a broad general fram work into which mathematical concepts fit in unified fashion. To assist you in teaching th new course, we have included many new feature described in this section of the *Teaching Guide*.

The unit-by-unit content of *STM 1* is given pages 2 and 3 of the student's book. The organ zation of each unit's work is shown in a panel the first page of the unit. An examination of t content of the panels will give you a good idea the work included in each unit.

The first page of each lesson in *STM 1* co tains important information for you and for t

ident. For example, at the beginning of each son, you will find a numeral that identifies the it and a numeral that identifies the lesson. At e bottom of the first page of the lesson, a very ief description of the content of the lesson is ovided for the teacher.

The numbering of the units and the lessons is quential throughout *STM 1*, *STM 2*, and *TM 3*. This numbering sequence was decided on partly to convey a feeling of continuity, but iefly to point up the fact that the mathematical ntent of the entire series is interdependent and terrelated. The content of each lesson and unit not isolated from the content of preceding les- ns and units, but is part of a continuing pro- am in mathematics.

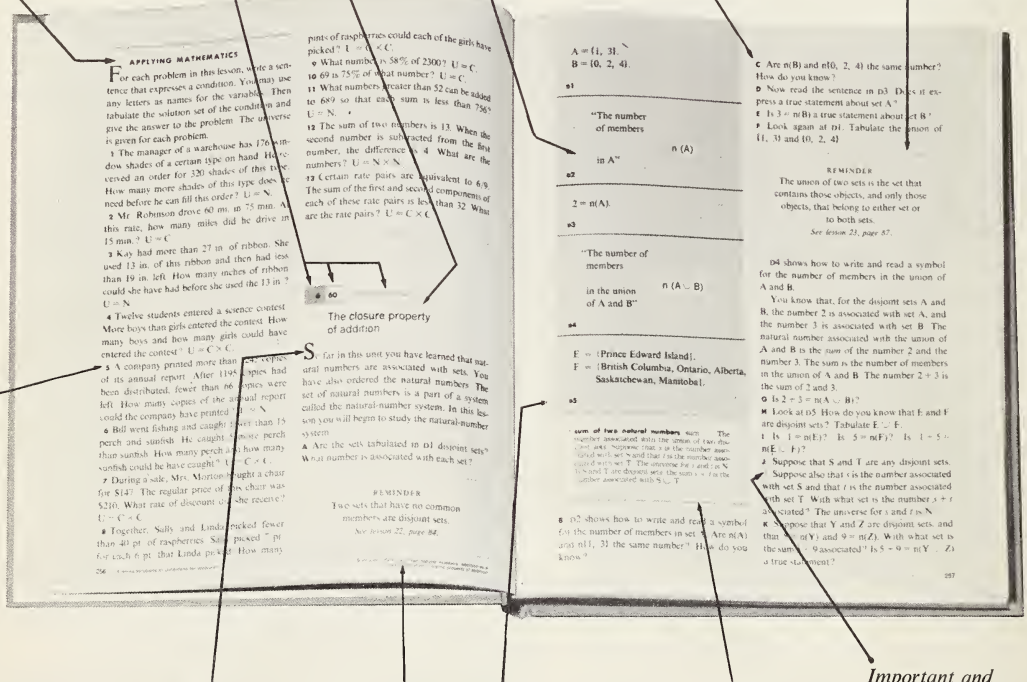
Each lesson begins with an introductory para- aph that gives the student a general idea of the sson content. Each lesson contains, at the end the developmental exercises, a paragraph that ummarizes the important ideas taught in the les- n. Each lesson includes two kinds of exercises, evelopmental exercises and "On your own" ex- cises. The developmental exercises (labeled with tters of the alphabet) are organized in "blocks" f work, each of which starts a new sequence of tters. Each such block is concerned with a group f closely related ideas or skills. For example, in sson 1, the first block of developmental exer- ises ends with exercise S on page 7. The second lock begins with A, also on page 7. A lesson ay contain only one block or as many as five locks of developmental exercises; most lessons ontain two or three blocks.

An important adjunct to teaching in *STM 1* is he displays, which are printed on a red back- ground. The displays contain examples, graphs, harts, and other materials to be used in con- unction with the developmental exercises. Each display is important to the development of a mathematical idea and is not inserted for the sake of color or design. Each display is either on the page that refers to it or on the facing page. When developmental exercises refer to a display that first appeared on a preceding page that has been turned over, the display is repeated. For example, the display labeled D4, which first appears on page 23 of *STM 1*, also appears on page 24. This arrangement permits the student to refer to a display without turning back to a preceding page.

Forty-five of the displays in *STM 1* show students how to read mathematical sentences and mathematical symbols. These "read it" displays are reproduced in the back of the student's book on pages 428, 429, and 430, thus permitting ready reference.

Notice the definition of *sum of two natural numbers* that appears on page 257 in *STM 1*. Many of the lessons include one or more such definitions. The definitions presented in the student's book are of two types, "dictionary" definitions and precise mathematical definitions. For example, the definition of *numeral* that appears on page 5 of *STM 1* is a dictionary definition; that is, the word is defined from an everyday point of view. However, the definition of *circle* that appears on page 46 is a mathematical definition; this definition is based on a mathematical point of view. All the mathematical definitions use symbolism and terminology that have been introduced previously in the course. The students should be encouraged to learn the meanings of new words that appear in *STM 1*, but they should never be required to memorize the definitions.

Italic type is used to call attention to key words the first time they are used in the student's book and for which definitions are included in the book. Italic type is also used to emphasize certain words that are not defined. Remember that, even in a formal development of mathematics, some terms are not defined. The term *variable*, for example, is undefined in *STM 1*, and this is also true



Lesson for practice on problem solving. Lessons labeled "Keeping skillful" provide practice on ideas and skills.

Lesson identification. Lesson 60, unit 6. New idea developed.

Lesson title.

Method of reading new symbolism.

Exercises that develop ideas or generalizations are labeled with letters.

Important idea needed here and previously tau Restatement of id and reference to original develop

Problems and exercises for independent work are labeled with numerals.

Paragraph to orient student with respect to content.

Examples and diagrams used in developmental exercises are shown in displays.

Important and difficult exercise. Correct response given in back of student's book.

Technical description of lesson content.

Dictionary-type definition of mathematical terms.

in textbooks for more advanced courses in mathematics. However, in *STM 1*, the properties of a variable are discussed and described. The understanding of these properties leads to an understanding of the idea of a variable.

Observe that certain developmental exercises in the student's book are labeled with a large red question mark. These exercises are designed to focus the attention of the student on the principal ideas presented in the lesson. When the student is confronted with one of these exercises, he should first give careful thought to formulating his response (or responses), and then he should compare his response with the response given in the back of the book (pages 431 through 445, *STM 1*).

Notice the "reminder" on page 257 of *STM 1*. Similar reminders appear throughout the book. They serve to refresh the memory of the student and also refer him back to the point in the text where a particular idea was developed. In this way, new ideas presented in *STM 1* are related to ideas presented earlier in the book.

The "On your own" exercises, included at the end of each lesson, are intended to be used as oral or written practice for the students. Each such exercise is labeled with a numeral, and the responses for each set of exercises appear in the *Response Verification Book* for *STM 1*.

"Exploring" headings identify two kinds of developmental lessons, "Exploring ideas" and "Exploring problems." Of the 97 lessons in *STM 1*, seventy-three are classified as "Exploring ideas." These lessons develop concepts that are new to the students, and any problem solving or computation that is included in these lessons is of secondary importance. The ideas of sets, set operations, rate pairs, number systems, and so on, are developed in lessons with the "Exploring ideas" heading.

In the nineteen "Exploring problems" lessons, the students apply the mathematics they have learned to the solution of verbal problems. After they have learned how to find solutions for simple conditions in one variable, they learn to solve problems that involve simple conditions in one variable. When they have learned how to find solutions for compound conditions in two variables, they learn to solve problems that involve compound conditions in two variables. Thus, after the students have studied certain sequences of mathematical ideas in the "Exploring ideas" lessons, they use these ideas to solve problems in the "Exploring problems" lessons. Notice that the problems are *not* used primarily to motivate or teach the mathematics; the problems are used as an application of the mathematics the students have studied.

The heading "Learning computation" identifies lessons that review the skills involved in computation. The first three such lessons, contained in unit 1, review the skills involved in multiplication and division of natural numbers. The two other lessons of this kind, contained in unit 7, review computation with mixed numerals and decimal fraction numerals.

Other headings serve to identify sets of exercises that provide maintenance of ideas and skills. The heading "Keeping skillful" identifies forty-three blocks of exercises that help to maintain mathematical ideas and skills. The first set of "Keeping skillful" exercises appears on page 60. This set contains exercises on computation, on the tabulation of solution sets of conditions, and on making graphs. An example of a "Keeping skillful" block that provides for the maintenance of geometric ideas is found on page 68.

Sets of problems that provide maintenance are identified by the heading "Applying mathematics." These sets include examples of types of problems that the student has previously studied. Terms such as *commission*, *discount*, *interest*, *principal*, and *rate of interest* are introduced in

these problems. These terms and their meanings are not presented in developmental lessons because they do not describe mathematical ideas. Rather, they are treated as what they are—business applications of mathematical ideas. For this reason they are included in the sets of "Applying Mathematics" problems.

The heading "Special challenge" identifies certain sets of exercises that may be used to provide enrichment for abler students. There are eleven such sets in *STM 1*. The answers for the "Special challenge" exercises are given on pages 256 through 259 of this *Teaching Guide*.

The heading "Checking up" is used for the tests that appear in the student's book. *STM 1* contains four types of tests: end-of-block, end-of-unit, cumulative, and inventory. There are sixteen groups of end-of-block tests. Each of these groups of tests follows a series of three through seven lessons. The end-of-block tests may be used to evaluate class progress or as the basis for tests you may want to construct on your own.

There are no end-of-block tests for the last group of lessons in each unit because the test items that cover concepts and skills developed in these final lessons are included in the end-of-unit tests. The end-of-unit tests are objective tests of the following types: completion, multiple-choice, true-false, matching, objective-response, and problem-solving. Each item in these tests is followed by a numeral within parentheses. This numeral identifies the page on which the idea tested by that item is developed. Thus, the end-of-unit tests may be used for review purposes as well as for evaluation of learning.

Cumulative tests follow each end-of-unit test except, of course, the one for unit 1. The cumulative tests contain items that test ideas from each of the preceding units, including the unit just completed. These tests are of the same types as those in the end-of-unit tests, and a page reference is furnished for each item.

Responses for items in all the tests are given under the heading "Checking up" in the *Response Verification Books for STM 1*.

On pages 232 through 248 of this *Teaching Guide*, other tests are provided, one for each unit in *STM 1* and an end-of-book test. You may find it desirable to reproduce these tests for use in evaluating each student's progress. These tests appear *only* in this *Teaching Guide*. Thus you have available for class use tests that are not in the student's book.

A picture on page 232 suggests a form in which you might want to type the tests for reproduction. The arrangement of the items in this suggested form takes into consideration convenience in scoring the responses. The suggestion is also given that a response strip be made to fit the response spaces that will enable you to use the same test papers year after year.

We hope that you will find the many new features in *STM 1* helpful as you teach this course.

Adapting method to individual differences

■ At the planning stage, the authors of the *Seeing Through Mathematics* program considered the recommendations of major study groups that are concerned with the improvement of the mathematics curriculum for the junior high school. They also gave careful thought to two questions: Could seventh-, eighth-, and ninth-grade students grasp and apply the basic mathematical ideas the authors wanted to present? Could teachers present this material without having special training?

To obtain answers to these two questions, the authors and the editorial staff prepared experi-

mental materials. These materials were then used in the *Scott, Foresman Experimental Mathematics Project*. Four hundred teachers taught these materials to approximately 20,000 students in 600 classrooms. The excellent results of this experimental project and the enthusiastic responses of teachers and students convinced both authors and staff that a modern mathematics program, using effective methods of presentation, could provide meaningful learning experiences for students at all levels of ability. One major outcome of the experimental project was the realization that students are able to learn much more mathematics than has been taught heretofore.

The authors and editors of *STM* believe that, by using what we know about the principles of learning, we can teach mathematics to the slow learner in an intellectually honest way. This does not mean that the slow learners are going to be mathematicians. Neither does it mean that all the abler students are going to be mathematicians. However, it does mean that, while students proceed at different rates and progress to different levels in the main stream of mathematics, they study essentially the same body of mathematical content. Some students may terminate their study of mathematics sooner than others, but the mathematics they learn has been achieved through understanding.

STM, which is an integrated mathematics program for grades 7, 8, and 9, helps in many ways to provide for individual differences. Its chief provision is to enable boys and girls to progress at their own rates of learning. This is accomplished in three ways: by the method of presentation, by provisions for practice and maintenance, and by the "packaging" of the books (*STM 1*, *STM 2*, *STM 3*).

The method of presentation is especially helpful in developing mathematical proficiency at different rates of learning. For example, the more than adequate number of developmental exercises assists in providing understanding for *all* the students. By complete use of these exercises, the slow learners can be led to an understanding of the concepts that are being taught. The abler student, who requires less time to arrive at a generalization, can omit some of these exercises.

The developmental method of presentation makes possible another provision for individual differences. Groups of students may proceed "on their own," using *STM 1* and the *Response Verification Books* that accompany *STM 1*. When used in this way, the *Response Verification Books* become vital learning aids for the student. For example, he might use the *Response Verification Books* to verify his responses to the developmental exercises, or he might use them to find the best response to a question. In either case, he should refer to the response only after he has given considerable thought to the exercise.

Another way that the abler students can use *STM 1* on their own is to give special attention to the questions identified by large red question marks. They might quickly read through all the developmental exercises, but pause and give serious thought to these particular questions. Since the responses for these special exercises are included in the student's book (pages 431 through 445), the students may check their responses without the aid of the *Response Verification Books*.

The procedure just discussed should be used with abler students only. You might assume the rôle of an "adviser" when they are using *STM 1* in this way. Encourage the students to be self-reliant, and suggest references that will provide enrichment material. Guard against holding abler students back, but check carefully to be sure that they do not omit any new or essential material.

The "Special challenge" exercises included in *STM 1* may be used as enrichment material. Abler students might work on these exercises

while the slow learners in the class receive needed remedial help. The responses for these exercises are included in this *Teaching Guide*.

Practice and maintenance are vital parts of any mathematics program that is designed to help meet individual needs. In *STM 1*, once a concept has been developed, the maintenance program in the book provides a means of sustaining that concept or skill. Abler students require less practice and maintenance than slow learners. You may provide for individual differences by your use of practice material. For abler students, many of the "On your own" and "Keeping skillful" exercises may be omitted. Suggestions for such omissions are included in the lesson notes in this *Teaching Guide*, but you should use your knowledge of the abilities of your students to help you make decisions that concern omissions.

In some schools, even though homogeneous grouping would be desirable because of the extremely wide range in student abilities, it simply is not possible to organize such grouping. In such cases, school policy may permit within-class ability grouping.

Operating two or more subgroups within a class undoubtedly presents difficulties, and yet such a system does lend itself to a number of different teaching procedures. One method is to use a different set of materials with each subgroup. For example, slow learners in one subgroup might use *Special Book B** while another subgroup is using *STM 1*. If all class members must use the same book, the groups can work on different lessons, each group proceeding at its own pace. This system permits extensive use of easy examples and concrete objects in working with the slow learners and avoids the danger of boring abler students.

Another technique is to make use of frequent, short periods of class study. Following class study, you can discuss the lesson (or part of the lesson) with a particular group. Also, when one group is engaged in class study, you can work with other special groups in the class, students who have been absent, or the like. Be careful not to neglect a group as you become interested in the progress of other groups. *STM 1* is particularly useful for the approach described above because the developmental exercises offer the student an opportunity to engage in the type of thinking that is essential if he is to understand the basic concepts. This technique can be used effectively with regular classes, too. In any case, students should be encouraged to develop self-reliance in using *STM 1*.

At times it may be advisable to have abler students work with slow learners. This plan should not be allowed to become a "police" system, however, and students must be carefully paired if the desired results are to be achieved. The slow learner will obviously benefit from this approach if he is paired with an abler student who can explain the material in a meaningful way. The abler student will also benefit, since explaining the material will certainly aid him in obtaining a clear picture of the subject matter.

No program in mathematics can, in all honesty, be said to provide for the individual needs of all students. It is still essential for the student to *want* to learn. When factors beyond the control of the teacher impede a student's progress, he will do as poorly with a program in modern mathematics as with a traditional program.

In the *STM* books, together with their *Response Verification Books*, we have tried to give you a number of different ways to provide for individual differences. Our major assumption concerning individual differences is that rate of progress and final destination should control the

* Maurice L. Hartung, Henry Van Engen, and Lois Knowles. *Seeing Through Arithmetic, Special Book B* (Toronto: W. J. Gage Limited, 1962).

provisions made for differences in rate of learning. This rate of progress and final destination must be along the major stream of mathematics, even though the goal may not be the same for all.

Adjusting Content to individual differences

For schools that employ ability grouping, the "packaging" of the *STM* program provides flexibility in the use of the books.

Special edition. This Teaching Guide has been prepared to accompany the special edition for which each course for grade 7, grade 8, and grade 9 is available in one complete volume of seven units each. Users of this special edition should refer to their provincial curriculum guides.

Standard edition. In addition, the *STM* program is available in the standard edition which consists of two volumes per grade. Each of these volumes contains all the material found in the special edition, including approximately one unit of additional material per volume.

Teaching procedures for Book 1

■ It is unrealistic to suppose that, for any given grade or any given materials, there is just one correct method of teaching mathematics. A variety of teaching procedures, based on the classroom experiences of teachers who have used the experimental materials of this program, are presented in this section.

As an individual you will find that certain procedures are more effective than others, and your selection of specific procedures will be influenced by many factors, such as the number of students in the class and their ability range. Your study of the content and your use of *STM 1* in the classroom, more than anything we may say, will point out the teaching procedures that are best for you.

Procedures used in presenting traditional materials are not always adaptable to the presentation of what we are calling modern mathematics. Since traditional materials are generally organized around social applications, the method of presentation is usually descriptive. This method gives the student facts about a topic and then asks him to work with problems related to the given facts. The topics covered are not necessarily related, and the tasks of motivating the students, helping them understand the basic ideas, and providing for individual differences are left, for the most part, to the teacher. With such materials, teaching procedures generally follow the "tell and do" method.

The authors of *STM* believe that mathematics is best learned as one subject whose parts reinforce one another. This belief is reflected in a unified mathematics program that is modern in content, in pedagogy, and in spirit. A continuous recognition of, and a rigorous adherence to, the principles of the psychology of learning has been maintained throughout the lessons. Because the

method of presentation is inductive rather than descriptive, more questions appear in these lessons than in traditional materials. These questions are designed to enable the students to understand and retain mathematical concepts. They also will introduce students to the enjoyment of meeting an intellectual challenge, and will help them experience the satisfaction of achieving success in the creative study of mathematics.

You can see, then, that *STM 1* is a carefully thought out departure from the traditional seventh-grade textbook. As a result, the teaching procedures for this book should be built around the idea of "guided discovery." Bear this in mind as you decide on a particular teaching procedure for a lesson. To help you arrive at a decision for a given lesson, first direct your attention to the developmental exercises labeled with letters in each lesson. As has been previously noted, the lessons are built around these exercises. They systematically guide the student in making discoveries and generalizations. After careful study of a lesson, you may decide that with a particular class you will not use all the developmental exercises in one class period. Since these exercises are grouped in "blocks" within each lesson, some of them can conveniently be postponed without damaging the continuity of the lesson. Remember that the authors did not intend that each lesson be completed in one class session.

The first, and perhaps most obvious, procedure is to use, in a class discussion, each of the developmental exercises in the order in which they appear in the book. This is a good procedure for the first week of the school year because the classes may not yet be well organized. Also, this procedure will help the students to get acquainted with the book.

Discussion of each developmental exercise is particularly useful if the exercises are not too numerous or too lengthy. However, extensive use of this procedure, especially in lessons that are rather long, conceivably could result in a monotonous presentation, and we know that you share the desire to avoid the deadly monotony that often results from doing the same things in the same way day after day. Consequently, plan to vary your daily teaching plans whenever it is possible.

In some lessons the students may find the mathematical ideas difficult to understand, and for these lessons it may be necessary to follow the developmental exercises with great care. For such lessons, a thorough "airing" of the ideas can be provided by an exercise-by-exercise discussion. During such discussions, encourage the students to give careful thought to each question. Avoid the "once-over-lightly" approach to ideas that are essential to the development of the lesson.

A second procedure that may be used is to develop the ideas presented in a lesson without centering attention on the developmental exercises in the book. You may wish to copy on the chalkboard the displays given in the lesson and use these to help develop the lesson. The use of an overhead projector, particularly in the lessons on graphs, is very helpful. A specific example of such use is suggested by D10 on page 92 of *STM 1*. This display could be copied on two sheets of clear acetate, the first showing graph A and the second showing graph B. Use the first "transparency" to show the solution set of the simple condition $x + 3 > 7$. Then use the second "transparency" to show the solution set of the simple condition $6 + x < 14$. Then use the two "transparencies" together to show the solution set of the compound condition $x + 3 > 7 \wedge 6 + x < 14$. This device is also useful in teaching some of the lessons dealing with nonmetric geometry. Following your development of the lesson, the students can refer to the developmental exercises in the book to clear up any questions they have. Also,

they can refer to the book if they have trouble when they work the "On your own" exercises.

When you develop the ideas presented in a lesson, the amount of time that students need to spend in reading the developmental exercises is reduced. But this approach requires you to be thoroughly familiar with the content of the lesson. You must guard against omissions of important ideas.

You may find it desirable to have the students read the developmental exercises before the lesson is discussed in class. This may be a homework assignment, or you may set aside the first part of the period for reading time. The objective here is to have the students attain general (rather than specific) familiarity with the content of the lesson. The ensuing class discussion can then proceed more smoothly and rapidly.

A third procedure is to have the students independently study the developmental exercises. You may even assign some of the developmental exercises in a lesson as a written assignment. But this must be done with considerable care, because the writing of responses to all the questions may require several hours. The *Response Verification Book* may be used by students to check their responses. This book is more than an answer book, however. For each question in the developmental exercises, it contains a response that is designed to help the student understand and appreciate the significance of his answer.

There are a number of circumstances that may make it desirable for students to study the material on their own. A student who is new to the class or one who has been absent for a period of time may follow a program of individual study to "catch up." With a grouped class of abler students, you may also find this approach useful in reducing the amount of time spent on class discussions. In a heterogeneous class, you may find that the abler students are bored by the detailed discussion needed to teach the slow learners. The abler students, in such a case, might use *STM 1* and the *Response Verification Books* in a program of individual study. For such a program, it is essential that the students have a response book available. You may find it desirable to have a class set, or at least several copies, of the *Response Verification Books* on hand for use in programs of individual study.

Responses for the questions in the developmental exercises that are identified by a large red question mark are included in the student's book on pages 431 through 445. These questions are designed to give the student deeper mathematical insight than he is able to get from less challenging questions. The responses to these questions are included in the student's book so that a student can verify each response before he proceeds with the next question. You may find that abler students can study some lessons effectively by reading through the developmental exercises and giving close attention only to the exercises identified by a large question mark. As your class becomes organized so that abler students can work more and more on their own, you will be able to devote much of your time to those students who need special assistance.

A program of individual study also may result in other benefits. Since, for the abler student, such an experience may be his first opportunity to work on his own, there is great potential in using this approach to provide for individual differences. However, guard against the student's merely "skimming" through the exercises and thus failing to achieve clear understanding of the basic ideas involved.

To vary a program of individual study, you may have students work in pairs or in small groups. On some occasions, abler students may be teamed with slow learners. If this procedure is handled with care, the advantage for the slow

learner is apparent, while the better student, forced to explain and interpret, gains a better understanding of basic ideas.

The "On your own" exercises also provide opportunities for individual work. You may use them as homework assignments, or you may use them for supervised work in class. Your students will profit more from class discussion of these exercises if this discussion is delayed until all the students have completed the exercises. As with the lessons, a set of "On your own" exercises is not necessarily intended as one day's assignment.

Most likely you will find it possible to use more than one of the procedures just discussed as you proceed with the lessons. You may also wish to organize your classes into subgroups and use a different teaching procedure with each group.

Before you teach any lesson, be sure to go through all the lessons in the unit and acquaint yourself, in a general way, with the concepts developed. Then carefully consider each lesson before you decide on a particular teaching procedure for the lesson. As you read a lesson, refer to the "Lesson notes" in this *Teaching Guide* for the necessary background information and specific suggestions for teaching the lesson. Note that, for most lessons, suggestions in the lesson notes help provide for individual differences.

Remember that you are the best source of ideas for making use of any textbook. As you use *STM 1*, keep a record of the teaching procedures that you find to be most effective. These ideas, especially if they are discussed and shared with fellow teachers, will contribute greatly to your school's program in mathematics.

Lesson notes

This part of the *Teaching Guide* contains lesson notes for the 97 lessons in the student's book. The lesson notes for each lesson are organized in this way:

Objectives

A paragraph in which the objectives of the lesson are briefly described. The objectives are stated in terms of goals for the student.

Content overview

A discussion of the content of the lesson. Background material of a mathematical nature to provide information useful in teaching the lesson.

Instruction notes

Suggested teaching procedures for developing the lesson. Emphasis is on teaching the whole class.

Provision for individual differences

Recommendations for meeting the needs of abler students and slow learners. Suggestions for omission of material to suit slow learners. Provision of enrichment material for abler students.

Sentences and placeholders

Objectives

■ The students are informally introduced to the ideas of set, open sentence, closed sentence, condition, statement, universe, and solution set.

Content overview

■ Readiness for several mathematical ideas is provided in this lesson. The precise meanings of the ideas introduced are presented in the lessons that follow, where each of these ideas is treated in detail, and the necessary definitions or descriptions are developed.

In mathematics, as well as in other subjects, a frequent source of difficulty is confusion between an idea and the language that expresses the idea. Throughout the student's book we help the student avoid this confusion by distinguishing carefully between actual objects and the names of the objects.

The words "object" and "idea" can be used in the same way. A *number* is an abstract idea, and we can say that it is an object you cannot see. A *numeral* is a symbol that is used as a name for a number.

The study of mathematics is more meaningful when we make a distinction between an object (or idea) and a name of the object. The object to which a name refers is the *referent* of the name. The terms numeral and number illustrate what we mean by a *name-referent* distinction. Numeral, in this case, is the name; number is the referent. We can write or speak a name of an idea, but we cannot write or speak the idea itself. Although the term referent is used here, it is not introduced in the student's book.

Another example of the name-referent distinction involves a placeholder as opposed to a variable. A *variable* is the referent of a placeholder. A *placeholder* is a symbol that is to be replaced by the name of an object.

An object may be a member of a group, or set, and each such object may have many names (for example, the *number* seven may be named by the *symbols* 7, $12 - 5$, $6 + 1$, VII, and so on). Remember that we use the word "object" to refer to either a physical object or an abstract idea. In mathematics, a member of a set usually is an abstract object, such as a number or a point.

A mathematical *sentence* is made up of symbols that can be written or spoken. The term sentence, like the term numeral, is on the name level. A mathematical sentence may be *open*, or it may be *closed*. An open sentence may be described as "a sentence that expresses an idea that is neither true nor false," and a closed sentence may be described as "a sentence that expresses an idea that is either true or false."

A sentence that contains a *placeholder*—a blank, a star, and so on—expresses an idea that is neither true nor false (an open sentence). After a replacement is made for the placeholder, the sentence expresses an idea that is either true or false (a closed sentence). A sentence cannot express an idea that is both true and false.

The name of a member of a set can be used as a replacement for the placeholder in an open sentence like "n is a number greater than 6." When such a replacement is made, a closed sentence like "8 is a number greater than 6" is obtained. In lesson 1, non-mathematical sentences are presented first, to acquaint the students with the concepts that will apply to mathematical sentences.

Notice that only the name of an object can be used to replace a placeholder in an open sentence. For example, suppose that the letter y in the sentence "y is 10 years old" is replaced by the word

"Mary." The sentence "Mary is 10 years old" is obtained. Mary herself is not the replacement for the letter. The letter *y* was replaced by the name "Mary." The referent of that name is the person who is Mary. Similarly, when the letter *x* in the sentence "*x* is less than 6" is replaced by "3" and the sentence "3 is less than 6" is obtained, a numeral (not the number itself) is the replacement for the placeholder. A number is an idea and cannot be written. A numeral, however, can be written because it is a name for a number.

There are several features in the student's book that are designed to help you present the material to the students. The mathematical ideas that are developed in the lesson are listed for you at the bottom of the first page of the lesson. The introductory paragraph in each lesson provides the student with a general idea of the lesson content.

Definitions are boxed and prominently displayed. See, for example, the definition of *numeral* on page 5. The word that is defined is printed in boldface type. The pronunciation of words is indicated in terms of the scheme used in the *Dictionary of Canadian English*.

All displays are highlighted in color and labeled consecutively for each lesson. A very important feature in the book is the repetition of a display when another reference is made to a display that appeared earlier. An example of this feature is D4 on pages 23 and 24.

Throughout the developmental exercises, a large red "question mark" is used to indicate thought-provoking queries. In the student's book, reference is made to this particular type of exercise in the paragraph above exercise O on page 6. The responses for these exercises are given in the back of the student's book because we think the student should have an opportunity to verify his response immediately. Students should be encouraged to give considerable thought to these exercises before referring to the responses.

The use of precise language will be helpful in teaching students who use this textbook. Of course, you should avoid demanding precision when this may cause confusion, but, in general, the students will learn to respect the careful use of language if you use and emphasize the use of precise language.

Instruction notes

- Begin the lesson by having the students read the title and the introductory paragraph.

As you discuss developmental exercises A through I, you may alter the questions in the text so that they are about your own school. After exercise C, for example, ask: "Is it sensible to write the name of your English teacher in the blank following 'Mathematics teacher'?" [No]

Following exercise D, write on the chalkboard the names students have thought of for the number 7. Include a variety, such as $3 + 4$, $8 - 1$, and $21 \div 3$. Ask: "Do you think you could list all the names for the number 7?" [No]

After discussing exercise H, ask the class: "Does the sentence 'My name is_____' express an idea that is either true or false?" [No] "Does the sentence 'My name is Thomas Edison' express an idea that is either true or false?" [Yes]

Continue with the developmental exercises on pages 5 and 6. Before beginning page 7, say: "Look at the four displays on page 7. Are the sentences in these displays more like the sentence in D3, or more like the sentences in D4?" [D3]

You may find it helpful to do exercise J on the chalkboard. Write "*s* is greater than 6 and less than 10." Erase the *s* and test each numeral from 1 through 10 in turn. List the replacements that make the sentence express a true idea.

After completing the exercises through N on page 8, have the students turn back to the sentences in D3 and D5 through D8 (pages 6 and 7) and identify the placeholder in each sentence.

You might have the students generalize what they have learned in the lesson and compare their generalizations with the summary that precedes the "On your own" exercises.

Provision for individual differences

■ As you teach the first few lessons, you will be observing the strengths and weaknesses of your students. You will find it helpful to give careful consideration to the responses that students formulate for the developmental exercises. In general, abler students answer questions with greater facility and in greater depth than slow learners. Also, slow learners require more time to phrase responses with adequate precision and clarity. You should expect most of your students to have some difficulty in verbalizing their responses during the early part of the year.

Strive to develop a positive attitude with regard to individual differences. Encourage abler students to work independently whenever possible. Help the slow learners experience a sense of achievement so that they will be more willing to "dig in" when faced with work that is more difficult later in the course.

Students can reinforce the ideas they learn in a lesson only by using the ideas in practice; so we suggest giving them daily written assignments. You should, though, avoid the error of making assignments that are long and tedious. The abler student becomes bored with "busy work," and the slow learner may be overwhelmed with an assignment that you consider routine.

You should carefully study the developmental exercises and the "On your own" exercises before teaching the lesson. You may consider it advisable to omit some of the exercises. From time to time we will, in this part of the lesson notes, offer suggestions for possible omissions and, at times, we will include additional material. However, when deciding whether to omit certain exercises or to include additional material, you should base your decision on the needs of the students. In lesson 1, for example, you might consider reducing the number of names to be used as replacements for the placeholders in exercises 6 through 9 of the "On your own" exercises. Similarly, some of the exercises from 14 through 19 might be omitted.

---

1 / 2 Exploring ideas / Pages 9-12

Sets and subsets

Objectives

■ The students learn what is meant by a set, a member of a set, and a subset of a given set. They also learn to describe and tabulate sets and to read tabulations of sets.

Content overview

■ The importance of the idea of set in the *STM* program has been discussed on pages 8 and 9 of this *Teaching Guide*. A *set* is described as "a definite collection of objects." The word "objects" refers to either objects from the physical world (apples, dollars, and so on) or to mathematical objects. In mathematics, a member of a set is usually an abstract object, such as a number, a number pair, a point, or a line. The phrase "definite collection" refers to a well-defined collection, which means that the membership of the collection is clearly understood. Thus, "the collection of good books" is not a well-defined set because the membership is debatable. However, "the collection of novels written by Sinclair Lewis" is a well-defined set.

The symbol { } is introduced on page 11. As the symbols are introduced in the student's book they are shown in enlarged form. The developmental exercises call the attention of the student to the symbol and describe its use.

Another feature of the book is illustrated in D2 on page 11. We call a display of this kind a "read it" because its function is to teach the student the correct way to read a symbol used in the book. The student should be encouraged to use correct terminology in reading symbols. You will find that the retention of ideas by students will be strengthened throughout the course by the consistent use of precise language.

In lesson 2 the students study two ways to talk about a set. The members of a set may have a characteristic in common. In such cases, a set can be *described* in terms of this characteristic. The natural numbers that are solutions of "$x + 3$ is less than 5" have a common characteristic; therefore, this set of numbers can be described as "the set of natural numbers such that the sum of each number and 3 is less than 5."

Not all sets can be described in this manner, however. The members of some sets are so dissimilar that a common characteristic is difficult to find. In such cases, the set can be identified by using a name for each member of the set; that is, by *tabulating* the set. For a set that has many members, it is not practical to name each member. For a set with infinitely many members, it is not possible to tabulate all the members. Tabulations of such sets as the set of natural numbers from 2 through 1000 or the set of natural numbers greater than 100 are discussed in lesson 6.

The relations among set, subset, and open sentence are informally introduced in this lesson. Ideas involving these terms are more fully developed in lessons 3 and 4.

From the definition of *subset*, it follows that a given set is a subset of itself. For example, each member of {a, b, c} is a member of {a, b, c}; therefore, {a, b, c} is considered to be a subset of itself. Also, a set that has no members, such as the set of women who have been Prime Minister of Canada, is considered a subset of any set.

Instruction notes

■ Before beginning lesson 2, briefly remind the class of what they learned in lesson 1.

After leading your class through a discussion of exercise C on page 9, ask: "Are you, the students in this class, a set?" [Yes] "Who are some of the persons in this set? What are some other sets to which you belong?" [The set of Boy Scouts or Girl Scouts, the set of persons in a family, the set of Canadian citizens, and so on] "What are some objects in the set of all musical instruments? What are some objects in the set of major-league hockey teams?"

After exercise K on page 10, ask: "Is the set of teachers in this school a subset of the set of students enrolled in this school?" [No] "Is each of you in this class also a member of the set of students enrolled in this school?" [Yes] "Is this class a subset of the set of students enrolled in this school?" [Yes] "Is the set of students enrolled in this school a subset of the set of students in this class?" [No]

Continue with the developmental exercises through exercise T on page 12. Then say: "You have been replacing n in a sentence by the numerals from 1 through 12. Is it sensible to replace n by the name of a book?" [No] "Is it sensible to replace n by the name of a city?" [No]

To summarize, ask a student to describe a set. Ask other students to name members of the set and to describe subsets of the given set. Then have a student tabulate a set at the chalkboard. Ask other students to read the tabulation, name members of the set, and tabulate various subsets.

Provision for individual differences

■ The abler students might consider the following items that are related to the ideas in the lesson.

1 One set is a subset of a second set, and the second set is a subset of the first set. What must be true about the two sets? [They are equal.]

2 "Equals" is a relation between numbers. Similarly, "is a subset of" is a relation between sets. If set A is a subset of set B, will set B also be a subset of set A? [Not unless they are the same set] If A is a subset of B and B is a subset of C, will A also be a subset of C? [Yes]

Sentences, statements, and conditions

Objectives

■ The students learn several important distinctions between a name of an object and the object itself: between closed sentence and statement; between open sentence and condition; between placeholder and variable.

Content overview

■ A name-referent distinction is helpful in understanding a program in modern mathematics. In a traditional program, the names, or symbols, were frequently confused with ideas. Focusing attention on symbols does not ensure understanding of ideas.

In lesson 1 the students learned that numbers are mathematical objects, or ideas, and that numbers are expressed by numerals.

A sentence that does not contain a placeholder is made up of symbols that express an idea. Since a sentence is a set of symbols, it can be neither true nor false, but the idea expressed by the sentence may be true or false. The sentences in D1 through D5 express statements. A *statement* is an idea that is either true or false. A statement is expressed by a *closed sentence;* for our purposes at present, this means a sentence that does not contain a placeholder.

Sentences such as those in D6, D7, and D8 (pages 14 and 15) express conditions. A *condition* is a requirement, and is neither true nor false. The requirement of the condition $x + 1 < 5$ is that, to obtain a true statement from the condition, the sum of the replacement for the variable and 1 must be less than 5. A condition is expressed by an *open sentence;* that is, a sentence that contains a placeholder.

Just as different numerals may express the same number, so different sentences may express the same idea. The sentences in the displays illustrate this fact.

Since a placeholder is a symbol in a sentence, it has a referent. The referent of a placeholder is a *variable.* A variable is an idea. A variable may be thought of as a kind of placeholder at the idea level. In the sentence "3 is less than x," the placeholder x may be replaced by the numeral "4." In the condition "3 is less than x," the variable x may be replaced by the number 4. As you can see, a placeholder (a symbol) is replaced by other symbols, but a variable (an idea) is replaced by other ideas (numbers or points).

Our description of variable as developed in *STM 1* differs from that of some other authors. In the past this term was poorly defined. At one point it was defined as a symbol. Unfortunately, many people followed this example. In seeking a more defensible position, we accept variable only at the idea level because we think that important terms should be defined at the idea level rather than at the symbol level.

Do not expect your students to use all the new mathematical vocabulary correctly at first. Appreciation of the need for precise language will come with mathematical maturity and exposure to proper usage of terms. You should use correct terminology, but you should not allow your concern with this to interfere with easy communication. You and your students will gradually adopt the more precise terminology that is characteristic of good teaching of mathematics.

Instruction notes

■ Discuss exercises A through M on page 13 with your class. Then ask: "Does each of the sentences in displays 1 through 4 express a statement?" [Yes. Each sentence expresses an idea that is either true or false.] "Do you think that all sentences express ideas that are either true or false?" Let the class talk about this for a short time. The students should understand that the answer to the question is "No." Continue with the text.

Following exercise E on page 14, ask: "In this lesson, how were you able to tell quickly if a sentence was open or closed?" [It was open if it contained a placeholder; it was closed if it did not contain a placeholder.] "Can the same condition be expressed by different sentences?" [Yes. Both sentences in D6 express the same condition.]

Continue with the developmental exercises to the end of the lesson. Write "7 is greater than x" on the chalkboard. Then ask: "What kind of sentence have I just written on the board?" [An open sentence] "What does this sentence express?" [A condition] "What is the x called in this sentence?" [A placeholder] "What is the x in the condition?" [A variable]

Then write "7 is greater than 5" on the board. Ask: "What kind of sentence have I written on the board this time?" [A closed sentence] "What does this sentence express?" [A statement] "Is a statement always either true or false?" [Yes] "Is this statement true, or is it false?" [True]

Provision for individual differences

■ Do not omit any of the developmental exercises for any of the students.

Abler students can further strengthen their understanding of the distinction between a symbol and what the symbol expresses by using the following activity. Tell them to use quotation marks when reference is specifically made to a symbol (as in paragraphs A, B, and C in the next column). Dictate these paragraphs to the students (without indicating quotation marks) and have them insert quotation marks in the proper places.

A If you subtract 3 from 43, you have 40, but if you remove "3" from "43," you have "4." If you add 3 to 40, you get 43. You can get either "43" or "34" by combining "3" and "4."

B 99 divided by 9 is 11, but "99" contains "9" just twice. So when you subtract eleven 9's from 99, the result is 0, but when two "9's" are removed from "99," the result is a blank space.

C "7" may be smaller than "4," but 7 is greater than 4. Likewise, ".007" is longer than "7," but .007 is less than 7.

Slow learners might use sentences like those below for practice in distinguishing symbols from ideas expressed by symbols. Dictate the sentences to them (without indicating quotation marks) and have them place quotation marks around each symbol that refers to a symbol itself.

1 "Al" is a nickname for Alvin.
2 "4" is a name of a number.
3 A "cat" is in "catsup."
4 4 is an even number.
5 "8" is a numeral for the number 8.
6 "5 + 6" is a name for 11.
7 "11" is a name for 5 + 6.

---

1 / 4 Exploring ideas / Pages 16-18

Sets, conditions, and equations

Objectives

■ The students become familiar with the universe for a variable, conditions for equality, equations, and solution sets of conditions. They learn that a statement can be obtained from a condition by replacing the variable by a member of a given set, or universe.

Content overview

■ Lesson 3 emphasized the distinction between a condition and a statement. In this lesson and in following lessons, the ideas of condition, set, and statement are related. Open sentences, tabulations, and closed sentences are used as a means of communicating ideas.

Since only the members of the *universe* can be used to replace a variable in a given condition, it is necessary to specify a universe for the variable. Each member of the universe either satisfies the condition or does not satisfy the condition. The number of members that satisfy a given condition depends on the universe as well as on the condition.

A universe can be a set of students, a set of letters of the alphabet, or a set of numbers. In lesson 12, where geometric conditions are considered, a set of points is used as a universe. In unit 3, a set of ordered pairs of numbers is used as a universe.

In many traditional programs, conditions for equality, true statements that involve the idea of equality, open sentences that contain the symbol for equality, and closed sentences that contain the symbol for equality are all identified as equations. Our definition of equation is more limiting.

Since an equation is considered an idea, students cannot write equations, but they can write closed sentences that express equations. An *equation*, then, is a true statement that involves the idea of equality. In mathematics, "equals" can be interpreted as meaning "the same as." The statement $5(6) = 30$ means that the number $5(6)$ is the same as the number 30. The symbol $5(6)$ and the symbol 30 name the same number. The statement $5(6) = 30$ includes the idea of equality and is a true statement; therefore, it is an equation. $0 + 5 = 0$ means that $0 + 5$ is the same number as 0. Even though this statement involves the idea of equality, it is not an equation, because it is false. The symbol "=" means the same thing in the sentence $5(6) = 30$ as it does in the sentence $0 + 5 = 0$. Whether the statement is true or not does not change the meaning of the symbol.

A *condition for equality* is simply a condition that involves a requirement of equality. The *solution set of a condition* is the set of those members of the universe that satisfy the condition. Each member of the solution set must be a member of the universe for the variable. Hence, a solution set always is a subset of the universe. Remember that a set is a subset of itself. When every member of the universe satisfies a condition, the solution set of that condition is still a subset of the universe. When no member of the universe satisfies the condition, the solution set is the empty set (symbolized by { }). Since the empty set is a subset of every set, it, too, is a subset of the universe.

In this lesson, the solution sets of conditions for equality are obtained by making replacements for the variables. In later lessons, computation is used to find the members of the solution set.

Instruction notes

■ At the beginning of the lesson, say: "Think about the sentence that you read a few days ago in lesson 1, 'x has a birthday in September.' Is this an open sentence or a closed sentence?" [Open] "Do you remember that we replaced the placeholder x in this sentence by the names of some students? After we replaced the symbol x by a name of a student, what kind of sentence did we obtain?" [Closed] "Did some of the closed sentences that we obtained express true statements, and did some express false statements?" [Yes] "Now, in lesson 4, you are going to learn how to obtain statements from a condition involving numbers."

Begin the developmental exercises. After exercise B on page 16, ask: "Would it make sense to replace x in $x+6 = 9$ by Toronto?" [No]

'Would it make sense to replace x by Asia?" [No]
'Would it make sense to replace x in this condition by any object other than a number?" [No]

Continue with the lesson through exercise I on page 16. Then say: "Turn back to page 6. What did you use as the universe for the variable in the condition expressed in D3?" [The students named in D2] "On page 7, what did you use as the universe for the variable in the condition expressed in D5?" [The letters of the alphabet] "What did you use as the universe for the variable in the condition expressed in D6?" [The numbers from 0 through 8] "To obtain a statement, must a variable in a condition have a universe from which replacements can be obtained?" [Yes] "If a condition involves numbers, is it sensible for the universe to be a set of numbers?" [Yes]

Proceed with the developmental exercises through H on page 17. Then ask: "Does the sentence $3 + n = 15$ express a statement?" [No. It expresses a condition.] "Does the sentence $3 + n = 15$ express an equation?" [No. Only a statement can be an equation.] "Is a condition ever an equation?" [No. A condition is not a statement; therefore, it cannot be an equation.]

Return to the developmental exercises. After discussing N on page 18, ask: "What does the $2x$ in the condition mean?" [That a replacement for x is multiplied by 2]

Provision for individual differences

■ For a class of abler students, you could use this lesson as a homework assignment and then discuss the lesson in class.

The students might also consider questions of this type:

1 Is the statement expressed by the sentence "Five is the same number as the sum of two and three" an equation? [Yes]

2 The statements expressed in the next column were obtained by replacing the variable in a condition by each member of the universe. What is the condition? [The condition is expressed as $2x + 6 = 4x$.] What is the universe? [{1, 2, 3, 4}] What is the solution set of the condition? [{3}]

$$2(1) + 6 = 4(1).$$
$$2(2) + 6 = 4(2).$$
$$2(3) + 6 = 4(3).$$
$$2(4) + 6 = 4(4).$$

For students who need further practice, make a list of sentences that express conditions for equality. Let the universe for each variable be the set of natural numbers from 1 through 20. Have the students tabulate the solution set of each condition.

Have slow learners write sentences that express equations. Have them also write sentences that express conditions for equality.

1 / 5 Exploring ideas / Pages 19-22

Sets, conditions, and inequalities

Objectives

■ The students learn to use the idea of equality to associate a set identified by one name with the same set identified by another name. They also learn to use the ideas of "less than," "greater than," and "not equal to" in conditions for inequality and in inequalities. The students then learn to find solution sets of conditions for inequality, using finite universes.

Content overview

■ The idea of equality in $5(6) = 30$ has basically the same meaning as the idea of equality in $K = \{5, 6, 7, 8, 9, 10\}$. As pointed out in the "Content overview" for lesson 4, $5(6) = 30$ means that the number $5(6)$ and the number 30 are the same. $K = \{5, 6, 7, 8, 9, 10\}$ means that K and

{5, 6, 7, 8, 9, 10} are the same set. Therefore, $K = \{5, 6, 7, 8, 9, 10\}$ is an equation rather than a condition for equality.

Remember that *conditions for inequality* are expressed by open sentences (sentences with placeholders). *Inequalities* are true statements that involve an idea of "not equal to" ("greater than," "less than," or simply "not equal to").

Instruction notes

■ To start the lesson, say: "In the last lesson you learned that the idea of equality is involved in an equation. Turn to page 7. Do the sentences in D6, D7, and D8 express the idea of equality?" [No] "On page 13, do the sentences in the displays express the idea of equality?" [No] "In this lesson you will learn more about sentences that do not express the idea of equality."

After discussing exercise G on page 19, say: "Throughout this book you will find that the universe is usually named by the capital letter U. Why does it make sense to use the letter U rather than some other letter?" [It is easy to remember that you are referring to the universe when you use the letter U.]

After discussing the paragraph immediately before exercise A on page 20, ask: "Why is it easier to use a symbol such as this instead of the words 'is greater than'?" [It is shorter and, therefore, easier to write. We introduce new symbols for ideas that we will often use because they are easier to use than written words.] "Remember that in the last lesson we said that $5(6) = 30$ is an equation because it is a true statement that involves the idea of equality. Suppose you replace the x in $x + 2 > 6$ by 5. What do you obtain?" [$5 + 2 > 6$] "Is $5 + 2 > 6$ a true statement?" [Yes] "Does $5 + 2 > 6$ involve the idea of equality?" [No] "Is $5 + 2 > 6$ an equation?" [No]

Before beginning exercise E on page 20, ask: "Do the symbols for the ideas of 'greater than' and 'less than' point to the numeral for the lesser number or to the numeral for the greater numbe when they are in sentences that express tru statements?" [The numeral for the lesser numbe

You may want to display in your classroom poster similar to the one suggested below. Yo will find a felt marking pen an excellent aid i making such charts.

IDEAS EXPRESSED BY SENTENCES

CONDITIONS

Conditions for equality expressed by open sentences	Conditions for inequality expressed by open sentences
$x+3=9.$	$m+3<8.$
$n-2=13.$	$d-2>7.$
$5y=35.$	$16+21 \neq w.$
$4k+8=24.$	$27>a+4.$

STATEMENTS

True statements expressed by closed sentences	False statements expressed by closed sentences
Equations:	$6+2=5.$
$3+2=5.$	$14(3)<30.$
$4(12)=48.$	$5+0 \neq 5.$
$3+5=7+1.$	$34>35+1.$
Inequalities:	
$3+2>4.$	
$16 \neq 1+5.$	
$42>5+3.$	

Provision for individual differences

■ Abler students should be able to complete th lesson with little difficulty.

With a class of slow learners you could, at th point, pause to think about the material pr sented in lessons 1 through 5. "On your own" e

rcises 19 through 36 (page 22) could be used at ıe chalkboard. As your students work with these xercises, have them review the vocabulary and ymbolism introduced in the first five lessons. lave them write sentences that express the statenents obtained from each condition, and then ave them tell whether the statement obtained is n inequality, an equation, or a false statement.

/ 6 Exploring ideas / Pages 22-26

Finite sets and infinite sets

Objectives

The students learn the meaning of a finite set ınd an infinite set. They learn how to tabulate inite sets, including those that have many mempers, as well as infinite sets. The students also earn to use an infinite set as the universe for a variable and as a solution set for conditions.

Content overview

We call the set of numbers {0, 1, 2, . . .} the set of *natural numbers*. In some books, 0 is not included in the set of natural numbers. However, many mathematicians do include it, and there are good reasons for doing so.

If the number of members in a set is a natural number, then the set is *finite*. Since 3 is a natural number and the number of members in {a, b, c} is 3, {a, b, c} is a finite set. The set that has no members is called the *empty set*. Since the natural number 0 is the number of members in the empty set, the empty set is finite.

For some sets, such as the set of human beings on the earth, it is impossible to know the natural number that is the number of members in the set. But since the number of members in this set is a natural number, the set is finite, even if you cannot determine the number. A similar argument may be used involving the number of grains of sand on a large beach or in a truckload. In such cases, no one is really concerned with the number of members in the set, but it is, nevertheless, a natural number.

Sets that are not finite are *infinite*. The set of natural numbers is an infinite set.

Finite sets may have many members; for example, the set of natural numbers from 2 through 1000 or the set of odd numbers that are less than 2000. In such cases it is inconvenient to tabulate the set by writing the name of each member.

Strictly speaking, "tabulate" means to list every member of a set. Thus {1, 2, 3, . . .} is not a tabulation in the ordinary sense. However, "tabulate" is commonly used with reference to sets when writing {1, 2, 3, . . .} or {1, 2, 3, . . ., 100}. We use "tabulate" in both of these ways throughout the text.

Notice that the symbol in D2 on page 22 is read in such a way that a description of the set is given. Be sure students understand that a symbol such as {0, 1, 2, . . ., 1000} is considered a tabulation of the set. Some tabulations may be read in more than one way. For example, the symbol {5, 10, 15, . . ., 3000} may be read as "The set of natural numbers from 5 through 3000 that are multiples of 5"; as "The set of natural numbers from 5 through 3000 that are divisible by 5 with a remainder of 0"; or as "The set whose members are the numbers 5, 10, 15, and so on through 3000."

In the case of infinite sets, it is impossible to write a name for each member in the tabulation. The symbol in D5 on page 23 is a tabulation of an infinite set.

Throughout the book, we call the set of natural numbers "N." We use the sentence "U = N" to express the fact that the universe for a variable is the set of natural numbers.

Many ideas are involved when an infinite set is used as the universe for a variable. Because the universe is infinite, it is not possible to actually make each member a replacement for the variable in a physical sense. It is, however, possible to think of such replacements. The solution set of a condition for equality may be found either by using computation or by making some replacements for the variable. The solution set of a condition for inequality may be found by using the solution set of a related condition for equality.

The solution set of a condition such as $x + 3 = 3 + x$ is the universe for the variable. The solution set of a condition such as $x > 6 + x$ is the empty set. Remember that the empty set has *no* members. Even in the cases in which the solution set is the universe or in which the solution set is the empty set, each member of the solution set is a member of the universe. Hence, the solution set is always a subset of the universe; and since the empty set is a subset of every set, it is also a subset of the universe.

Only solution sets that can be tabulated are considered in this lesson. Conditions such as $x + 20 \neq 45$ when the universe for x is N are not considered. The best way to talk about the solution set of such a condition is to describe it.

Instruction notes

■ Before the paragraph that follows exercise A on page 22, ask: "How long would it take you to list the numerals from 0 through 1000?" [About fifteen minutes] After the paragraph that follows exercise A, ask: "How many members are named before the three dots?" [3] "When making this kind of tabulation, first you list the names of the first three members of the set. Why is it also important to name the greatest member of the set?" [To show that the set is finite and to indicate the greatest number in the set] "In what way is this kind of tabulation different from the kind you learned before?" [The order in which you list the names of the members of the set is important. When every member of the set is named, the order of listing is not important.]

Following the paragraph after exercise K on page 23, ask: "For any natural number named, can you name a greater number?" [Yes. You can add 1 to the number to obtain a greater number.]

Proceed with the developmental exercises through Q on page 25. Then ask: "What condition can you use to help find the solution set of $3x < 45$?" [$3x = 45$.] "If the universe for x is the set of natural numbers, what is the solution of $3x = 45$?" [15] "What are the solutions of $3x < 45$?" [0 through 14] "What is the solution set of $3x > 45$?" [{16, 17, 18, . . .}]

Continue with the exercises through M on page 25. Then say: "Remember that a set is a subset of a second set if every member of the first set is contained in the second set. What is the universe for exercise M?" [The set of students in the class] "Since there are no members in the empty set, it does not contradict the definition of subset to say 'The empty set is a subset of the set of students in this class.' Most mathematicians agree that the empty set should be considered a subset of every set. When the solution set of a condition is the empty set, is that solution set a subset of the universe?" [Yes, because it does not contradict the definition of subset]

Provision for individual differences

■ For students who are interested in very large numbers, you may do the following: Rule a duplicating-machine stencil into eighty 1-inch squares. Have the students take turns in making 100 dots in each square. Run off 125 copies so that you have 1,000,000 dots altogether. Tack the copies to your bulletin board, or tape them to a wall, so that your students may actually see one million dots. Ask the students to find out about a googol (10^{100}, a numeral for which is a "one" followed by 100 "zeros"). Have them find out about

googolplex ($10^{10^{100}}$, a numeral for which is a one" followed by a googol of "zeros").

Abler students might consider the following tems that are related to the work in the lesson:

1 Find an example in the lesson in which

a) both the universe and the solution set are finite sets. [The conditions expressed in D9 or D11]

b) both the universe and the solution set are infinite sets. [$x + 4 > 9$ in D6 or D7; $7 + m > 10$ in D8; $x + 3 = 3 + x$ in D10]

c) the universe is infinite, but the solution set is finite. [$x + 4 < 9$ or $x + 4 = 9$ in D6 or D7; $7 + m = 10$ or $7 + m < 10$ in D8; $1 + x = x$ in D12]

2 Can the solution set of a condition ever be an nfinite set when the universe for the variable is a inite set? Explain your answer. [No. The solution set can consist of members of the universe only.]

3 Tabulate any infinite set. Tabulate a second infinite set that is a subset of the first set. Then tabulate a finite set that is a subset of the first set. [An example is {0, 1, 2, . . .}; {0, 2, 4, . . .}; {0, 4, 8, 12}.]

4 Does every finite set have at least one finite subset? [Yes. Any set is a subset of itself. Also, the empty set is a finite subset of every set.] Does every infinite set have at least one infinite subset? [Yes. For example, a subset of {0, 5, 10, . . .} is {5, 10, 15, . . .}.]

Slow learners may have some difficulty in finding the solution sets for exercises 7 through 18 in the "On your own" section. Have them state in their own words what each condition requires. For each condition for inequality, have them state the related condition for equality. Have them first tabulate the solution set of the condition for equality, and then try numbers greater than and numbers less than the numbers in that solution set as replacements for the variable in the condition for inequality. As suggested in the notes for the first lesson, all work with slow learners will require that you proceed at a slower pace than with other classes.

1 / 7 Exploring ideas / Pages 26-28

Standard description of sets

Objectives

■ The students learn to describe a solution set of a condition by using a standard description.

Content overview

■ A standard description of a solution set is evolved in the lesson by first tabulating the set and then giving a verbal description of the set. The students will use braces and an open sentence to write a *standard description* throughout the remainder of the W. J. Gage Basic Mathematics Program.

Notice that, when the standard description is used, the universe for x is specified (see D7). A symbol like $\{x | x + 4 < 7\}$ does not actually describe a set unless the universe for the variable x is specified.

Observe the use of the words "all" and "each" in the descriptions in D2, D3, D4, and D6. We use "all" because we include every member of the universe for the variable that satisfies the condition. We use "each" because each member of the solution set of the condition satisfies the condition upon replacement of the variable. However, in reading the standard description, the word "each" is omitted.

It may seem cumbersome to have the students read "The set whose members are all x that satisfy the condition that x is less than six," as indicated in D6. As discussed previously, however,

precision of language at this point will be helpful in future lessons. In time, some of the terminology is abbreviated. It is important that pupils have a thorough understanding of the idea before "short cuts" in the language are employed.

Instruction notes

■ To begin this lesson, write "U = N. $x + 2 \neq 7$. {0, 1, 2, 3, 4, 6, . . .}" on the chalkboard. Ask: "What is the universe for the variable in the condition $x + 2 \neq 7$?" [The set of natural numbers] "Does this tabulation indicate the solution set of this condition?" [No. The meaning is not clear.] "Might this tabulation indicate the set of natural numbers except 5 and all multiples of 5?" [This could be what it means, but you cannot really be sure that is what it means.] "In this lesson you will learn a new way to indicate the solution set of a condition. You can use this new way for a condition like $x + 2 \neq 7$ and also for the conditions you have studied so far."

Begin the developmental exercises. After the paragraphs following F on page 27, ask: "Why is it important to think of *all* natural numbers that satisfy the condition?" [A solution set includes *every* member of the universe that satisfies the condition.] "Why is it important to think that *each* solution satisfies the condition?" [A solution set does not include any member of the universe that does not satisfy the condition.]

Your students may experience difficulty in thinking of the set described by a standard description. If so, you might have them read a standard description as "the set of all members of the universe so that each of these members gives a true statement upon replacement for the variable." If your students experience difficulty with the phrase (in exercise G) "all x such that . . .," point out that this means the same as "all x so that . . .," or use similar terminology until you are satisfied that they have achieved a clear understanding of the use of a standard description.

After exercise N on page 28, write "U = N $\{x | x + 3 > 3\}$" on the chalkboard. Ask: "What is the tabulation of this solution set?" [{1, 2, 3, . . .}] Write "U = N. {0, 1, 2, . . ., 32}." Say: "Now think of the set {0, 1, 2, . . ., 32}. What is a standard description of this set?" [$\{x | x < 33\}$, $\{x | x + 2 < 35\}$, and so on]

Now ask: "How would you indicate the solution set for the condition $x + 2 \neq 7$ that we talked about when we began the lesson?" [$\{x | x + 2 \neq 7\}$ or $\{x | x \neq 5\}$]

Provision for individual differences

■ Abler students might practice writing more than one standard description for the same solution set. For example, given the description $\{x | x = 1\}$ when U = N, they might write "$\{x | x = 1\} = \{x | x + 2 = 3\}$."

Slow learners could be given other tabulations and verbal descriptions of solution sets for which they could write standard descriptions.

Have students who need further practice write standard descriptions of solution sets of conditions that were studied in earlier lessons.

---

1 / Checking up / Page 28

End-of-block tests

■ This is the first group of the end-of-block tests which occur throughout *STM*. These tests occur under the heading "Checking up," as do all groups of tests in the student's book.

End-of-block tests are designed to test the students' ability to use the mathematical skills developed in the lessons that immediately precede the tests. Each of these tests covers from three through seven lessons. End-of-block tests are not designed to test the students' ability to verbalize concepts.

You may use these tests to help evaluate students' learning. If you want students to take a test exactly as it appears in the book, simply have them open their books to a specific page and write their responses on their papers. You can have the students write responses to all the items in the tests, or you can specify certain items that you want them to answer. Answers to all test items in the student's book are given in the *Response Verification Books* for *STM 1*.

You might wish to use an end-of-block test in the student's book as a homework assignment to review the lessons just covered. If so, tell students to restudy any topics they cannot cope with successfully.

You may, of course, prefer to construct your own test for evaluation of the student's learning. Such a test may be based on the types of exercises given in the end-of-block tests.

There are several ways in which you can administer a test that you prepare yourself. You can, for example, read the test to the students and have them write it on their papers and answer each item. As a rule, this technique requires a great deal of class time.

You can write the test on the chalkboard and have the students copy the test from the board and answer each item. This technique also requires much class time. Another alternative is to have the students write only the responses on their papers.

The most efficient way is to duplicate the test so that every student in the class may have a copy. The students can indicate their responses on the test paper, or you can supply each student with a separate answer sheet.

Specific suggestions for conducting a testing program are as follows:

1 Give tests frequently. Brief tests should be given at intervals of five through ten class meetings.

2 When you plan to give a test, give the students at least one day's advance notice. Explain in detail the ideas or skills that will be covered by the test.

3 Give close attention to the time needed to complete a test. Plan a test and the class time to be devoted to the test so that all students will be assured of ample time to complete the test.

4 Score the tests and return them to the students as soon as possible.

5 Discuss with the students the items answered incorrectly most frequently, and reteach specific concepts or skills, if necessary.

Use of these suggestions should help to make your testing program effective.

---

1 / 8 Exploring ideas / Pages 29-32

Sets of points

Objectives

■ The students study intuitively certain ideas about point, space, line, and plane.

Content overview

■ Lesson 8 is the first of five lessons on nonmetric geometry in unit 1. Nonmetric geometry is geometry that does not involve measurements.

The ideas about sets developed earlier are now related to sets whose members are *points*. The ideas of set, set membership, subset, and solution set are a few of the ideas that serve to integrate arithmetic, algebra, and geometry.

No formal definition is given for a point. In mathematics, some ideas are introduced by means of partial descriptions. We say that such ideas are undefined.

In most studies of geometry, the idea of point as well as the idea of line is undefined. If this pro-

cedure were not followed, a first idea might be defined by using a second idea, while the second idea might be defined by using the first idea. For example, consider the definitions given below:

1 Geometric figure: A set of points

2 Point: A member of a geometric figure

Such definitions are known as "circular" definitions. In mathematics we seek to avoid using circular definitions.

The students study fundamental ideas of geometry by seeing representations of these ideas and by discovering their characteristics. The tip of a pencil and a dot on a piece of paper each suggest the idea of a point. However, neither the tip of the pencil nor the dot is a point. A point has no size; a point has position, or location. A dot, on the contrary, may be large or small; a dot does have size.

A *line* is another mathematical idea that we choose to leave undefined. Unless otherwise noted, "line" means "straight line." "Part of a line" is described in exercise D on page 29.

Notice that the idea of betweenness for points is developed at an intuitive level in D1. This idea is more fully developed in lesson 10. Picture E in D1 represents a geometric figure.

The ideas presented in this lesson are basic to an understanding of geometry and are crucial to future work with geometry. Some new words are introduced, but their number is kept to a minimum.

The idea of space is introduced before the ideas of line and plane. The set of all points is called *space*. Dust particles in the earth's atmosphere suggest what is meant by the set of all points. Even though this set of dust particles is finite, it would be difficult to determine the number of members in the set. The set of all points in space is an infinite set.

A ray of light and a wire stretched tautly are used to suggest a line. Again, neither the ray of light nor the wire *is* a line. Each has a certain thickness as well as a beginning and an end. A line has no thickness, no beginning, and no end.

A line is considered an infinite set of points for two reasons: (1) A line extends endlessly in opposite directions, and (2) between any two points in a line, there are infinitely many points. A line is considered a subset of space, and each point in a line is a member of the set of all possible points. Since both a line and space are infinite sets, the idea that a line is a subset of space may seem unusual. Recall, however, that the set of natural numbers greater than five is a subset of the set of natural numbers, even though both sets are infinite.

A *plane* is another geometric idea. A set of dust particles illuminated by light directed into a room through a slightly opened door suggests what is meant by a plane. The surface of a flat wall also suggests a plane. Hence, a plane is a geometric figure and is a subset of space.

When an object is a member of a set, we say that the object is "contained in" or is "in" the set. Thus, a point is *in*, not *on*, a line (or infinitely many lines). A point is *in* a plane (or infinitely many planes), and a point is *in* space. When one set is a subset of another set, we say that the subset is "included in" the other set. Lines and planes are included in space, and lines are included in planes.

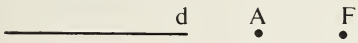
The first "reminder" appears on page 30 of the student's book. We use these reminders throughout the book to refresh the student's memory and to refer him to the point in the book where a particular idea is developed.

Instruction notes

■ You might begin by saying, "Before you came to this class, you had learned something about such figures as angles, triangles, squares, and circles. These figures are studied in the part of mathematics known as geometry. You will be studying

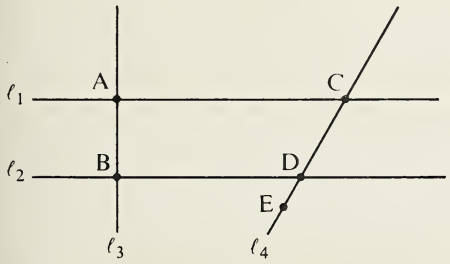
eometry again this year, beginning with this les-
on. You will find that the ideas you have learned
bout sets are also used in geometry."

Discuss the developmental exercises through J
n page 30. Then say: "You know that a set of
umbers has one or more subsets. Can a set of
oints have a subset?" [Yes] "What are three
ubsets of the set of points tabulated in exer-
ise J?" [{A, B, C}, {B, D}, { }, and so on]



Continue the exercises through H on page 31. Then make a sketch on the chalkboard that is similar to the one above. Say: "You see that the dots for points A and F are not included in the picture of the part of line d. You know that lines extend endlessly. Could points A and F be in line d?" [Yes] "We have said that pictures of lines are incomplete. Is it physically possible to represent a complete line?" [No. Physical objects do not extend endlessly.]

Proceed with exercises I and J on page 31. Sketch the following on the chalkboard:



Say: "Tabulate the set of points represented by dots in this picture." [{A, B, C, D, E}] "Tabulate the set of lines that are represented." [{ℓ_1, ℓ_2, ℓ_3, ℓ_4}] "Which members of the set of points that you tabulated are contained in ℓ_1?" [Points A and C] "Which members of the set of points that you tabulated are contained in ℓ_4?" [Points C, D, and E] "Tabulate the set of lines that are represented and contain point B." [{ℓ_2, ℓ_3}] "Is this set a subset of the set of lines suggested in the picture?" [Yes]

After the developmental exercises have been completed, see that each student has a small, circular piece of cardboard or a penny to represent a point, a pencil to represent a line, and a sheet of paper or tagboard to represent a plane. Ask your students to hold the penny and the pencil so that they represent a point in a line and then a point not in a line. Have them hold the penny and the paper to represent a point in a plane and then a point not in a plane. Have them use the pencil and paper to suggest a line included in a plane and then a line not included in a plane. Ask, "Can a line be contained in a point?" [No] "Can a point be contained in a line?" [Yes] "Can a line be a subset of a plane?" [Yes] "Can a plane be a subset of a line?" [No]

Provision for individual differences

■ Because of the newness and the importance of the ideas introduced in this lesson, all the developmental exercises and all the "On your own" exercises should be used for both the abler students and the slow learners.

The abler students might carry out some of the activities in the pamphlet *Paper Folding for the Mathematics Class*.* Many of the activities mentioned in this pamphlet emphasize some of the basic notions about points, lines, and planes that are discussed in this lesson and in the following lessons.

The slow learners will benefit from identifying physical objects in the classroom that suggest geometric figures.

* Donovan A. Johnson, *Paper Folding for the Mathematics Class* (Washington, D.C.: National Council of Teachers of Mathematics, 1957), 32 pp.

1 / 9 Exploring ideas / Pages 33-36

Relating points, lines, and planes

Objectives

■ The students learn to relate points, lines, and planes by means of properties.

Content overview

■ To relate geometric figures, some properties must be assumed. These assumed properties are then used to prove other properties of geometric figures. The properties expressed below, which are developed in this lesson, are some of the assumptions about geometric figures that are a part of the geometry we are developing.

1 For any two points, there is exactly one line that contains both of these points.

2 If a line contains two points of a plane, the line is included in the plane.

3 A given line is included in infinitely many planes.

4 There is exactly one plane that includes a given line and contains a point not in the line.

5 There is exactly one plane that contains three noncollinear points.

6 The set of all lines, each of which contains a given point, is an infinite set.

Property 6 is not listed in the text as a property, but it is developed in exercises D and E on page 33. When we say the property is "developed," we mean that the student is led to understand the meaning of the property. Many of the properties discussed are assumed and are not proved.

You will notice that none of the above properties involves measurement of any kind. Remember that the geometry we are developing at this time is nonmetric; that is, it is not based on measurement. Some fundamental ideas of metric geometry are developed in unit 8.

Notice that, throughout the *STM* program, when we ask the students to "make a sketch," we expect them to make a freehand drawing. When we ask them to "draw or construct," we expect them to use a straightedge, compasses, and/or a protractor.

Instruction notes

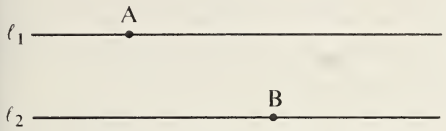
■ As you continue with the lessons on geometry, you might find it worth while to have each student compile a list of geometric figures, together with appropriate properties, definitions, and so on. An example of a figure that would be in the list is "line." Some characteristics of a line are that it is (1) a mathematical idea, (2) an infinite set of points, (3) a subset of space, and (4) a subset of infinitely many planes.

Because many geometric properties are developed, you may wish to spend two days on this lesson. If so, on the first day, discuss developmental exercises A through Q on pages 33 and 34 and assign "On your own" exercises 1 through 9.

After exercise E on page 33, say: "Hold the eraser end of your pencil so that it suggests another line that contains point C in D2. Now hold your pencil so that it suggests still another line that contains point C. In how many ways can you hold your pencil so that it suggests a line that contains point C?" [In infinitely many ways]

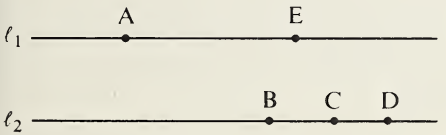
Continue with the exercises through K on page 33. Ask: "When you say two points determine one line, do you also mean that only one line contains the two points?" [Yes] "Look again at D2. What are two other names for ℓ_1?" [Line CD and line DC] "Since line CD and line DC name the same line, can you say that line CD equals line DC?" [Yes] "What are two names for ℓ_1 represented in D1?" [Line AB and line BA] "What are two names for the line represented in D3?" [Line FE and line EF]

Continue with the developmental exercises through P on page 34. On the chalkboard, sketch the figures below:



Say: "I have represented two lines on the chalkboard, and I have shown dots for two points. Think about these questions: 'Could you sketch a line AB? Are points A and B collinear?' " [Yes. Even though line AB is not represented, there is a line AB that contains points A and B. Therefore, any pair of points is collinear.]

After exercise Q on page 34, add dots for points C, D, and E to the sketches of lines 1 and 2, as shown below:



Say: "Explain why points A, B, and E are noncollinear and points B, C, and D are collinear." [No line contains points A, B, and E. Points B, C, and D are in ℓ_2.] "Are points D and E collinear?" [Yes. Although $\overleftrightarrow{DE}$ is not represented, it contains points D and E.] "In D4 on page 33, what sets of three noncollinear points are pictured?" [R, S, T; R, S, U; T, U, S; T, U, R]

You may want to review the content of this lesson up to this point and assign the first nine "On your own" exercises.

On the second day, continue with exercises A through D on page 34. Then say: "Make two dots on a piece of paper to represent points. Hold the paper flat so that it suggests a plane. Place your pencil on the paper so that the pencil touches the two dots. Does the pencil appear to touch the paper at many places between the two dots?" [Yes] "If two points in a line are in a given plane, is the entire line included in the plane?" [Yes] Finish this set of exercises.

Before beginning exercise A on page 35, set your classroom door in a half-open position. Point to the inner vertical edge of the door on the hinged side. Ask: "Does this edge of the door suggest a line?" [Yes] "Does the flat surface of the door facing you suggest a plane?" [Yes] "Is this line included in this plane?" [Yes] Close the door a few inches. "Does the door now suggest another plane?" [Yes] "Is the line also included in this plane?" [Yes] "Can a line be included in more than one plane?" [Yes] Place the door in a few more positions, and ask if the line is included in the plane suggested by each position. "Is a line included in infinitely many planes?" [Yes]

To illustrate the property developed in exercises F through H on page 35, you may use two pencils and a piece of tagboard. Using the length of one pencil and the tip of the other, illustrate the property developed here. Hold the pencils in several different positions to show that the tagboard fits flat against the pencils in any position that they are held.

For exercises I through M on page 35, have on hand a board with three nails driven into it to represent three noncollinear points. Place a piece of tagboard on two of the nails and show that it can be moved to many positions while still touching the heads of the two nails. Ask: "How many planes do you think contain these two points?" [Infinitely many] Place the tagboard on all three nail heads. "How many planes contain these three points?" [One] "Do three noncollinear points determine one plane?" [Yes] "Suppose I were to hammer another nail into this board. Would the point represented by the head of the

nail necessarily be in this plane?" [No] "Can you explain why three-legged stools and tripods always stand flat?" [The ends of the three legs suggest three noncollinear points. Three noncollinear points are contained in only one plane.]

Complete the remaining developmental exercises. Review each of the properties studied in this lesson, if necessary.

Provision for individual differences

■ You could duplicate the following questions for the abler students to work on while you give help to the slow learners:

1 In "On your own" exercises 9 and 10, you were asked how many lines are determined by four points and how many lines are determined by five points. In each case you knew that no three of the points were collinear and that all the points were in the same plane.

(a) How many lines are determined by six points, no three of which are collinear? [15]

(b) How many lines are determined by seven points, no three of which are collinear? [21]

(c) How many lines are determined by eight points, no three of which are collinear? [28]

(d) Think of your answers to these last questions. Without making a sketch, can you tell how many lines are determined by ten points, no three of which are collinear? [45 lines. 2 points determine 1 line; 3 points determine $1 + 2$, or 3 lines; 4 points determine $1 + 2 + 3$, or 6 lines; 5 points determine $1 + 2 + 3 + 4$, or 10 lines; 10 points determine $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$, or 45 lines; and so on.]

2 You have learned that the names of two points in a line can be used to name the line.

(a) In how many ways can you name a line if you know the names of three points in the line? [3×2, or 6]

(b) In how many ways can you name a line if you know the names of four points in the line? [4×3, or 12]

3 You have learned that the names of three noncollinear points in a plane can be used to name the plane.

(a) In how many ways can you name a plane, if you know the names of four points in the plane, no three of which are collinear? [$4 \times 3 \times 2$, or 24]

(b) In how many ways can you name a plane, if you know the names of five points in the plane, no three of which are collinear? [$5 \times 4 \times 3$, or 60]

Have the abler students make models of the geometric figures shown in the displays in the lesson. You will be able to use these models with your slow learners. Plywood, thumbtacks, and string are all they will need.

With slow learners, use many examples to demonstrate the properties discussed in the lesson. Involve the students in the demonstrations as much as possible. If you have not used the developmental exercises in teaching the lesson, some of these exercises could be assigned to the slow learners.

---

1 / 10 Exploring ideas / Pages 37-40

Segments and congruent segments

Objectives

■ The students use properties that relate points, lines, and planes in studying subsets of lines. They learn how these subsets are related to each other.

Content overview

■ The idea of *betweenness* for points is introduced in preparation for the definition of line segment. The idea of betweenness is important in a careful development of geometry. In this lesson an informal development is given as an introduction. Betweenness for numbers is discussed in a later

esson. The three properties of betweenness for oints given below are developed in the lesson.

1 If one point is between two other points, then he three points are collinear.

2 For every three collinear points, just one of the oints is between the other two.

3 For any two points, there is at least one point etween them.

Notice that each property is an assertion consisting of two parts. The first part, called the ypothesis, is what is to be taken as known. The econd part, called the conclusion, is what can be oncluded from what is known. The first property given above tells you that, if you know that one point is between two other points, then you can conclude that the three points are collinear. In order to apply this property, you must have a situation in which the hypothesis is satisfied. You must know that one point is between two other points before you can use this property to conclude that the three points are collinear.

The second property tells you that, if you know that three points are collinear, you can conclude that one, and only one, of these points is between the other two. Notice that the hypothesis of property 2 is the same as the conclusion of property 1. However, the conclusion of property 2 is somewhat different from the hypothesis of property 1.

When the hypothesis and the conclusion of a statement are interchanged, then the new statement obtained is said to be a converse of the original statement. Property 2 is not a converse of property 1. Below are two statements and a converse of each statement:

1 If an animal is a dog, then it walks on four legs.

2 If it walks on four legs, then an animal is a dog.

3 If two sets are equal, then the two sets have the same members.

4 If two sets have the same members, then the two sets are equal.

Statement 2 is a converse of statement 1, and vice versa. Statement 4 is a converse of statement 3, and vice versa. Although statement 1 is true, statement 2 may be false. Both statements 3 and 4 are true. You may, or may not, wish to discuss the idea of a converse with your students. This idea is developed further in lesson 103. It is discussed here more for your benefit than for the benefit of the students. Should you choose to discuss the idea with your students, you obviously will not wish to belabor the point at this time.

In answering some of the exercises, the students will be relying on their intuitive understanding of betweenness. However, once a property is formally stated, the students are expected to use the property when they explain a "yes-or-no" answer. For example, the students will give an affirmative answer to each of the questions in exercise B, page 37, on the basis of their intuitive understanding of the idea of betweenness. But their answer to exercise D, page 37, should be based on the formally stated property: "If one point is between two other points, then the three points are collinear." Similar comments could be made regarding the answers to exercises I and K on the same page.

To answer exercise Q on page 38, we must assume a fourth property of betweenness that is not formally stated in the lesson. This property is: "If a point F is between A and D, and D is between A and B, then point F is also between A and B." This assumption, combined with the property "For any two points, there is at least one point between them," enables us to explain, in answering exercise T on page 38, why there must be infinitely many points between two given points.

A formal proof is suggested below. An *informal* discussion along these lines may be used with students.

Given: Points A and B.

To prove: There are infinitely many points between A and B.

1 There is at least one point F between A and B. (For any two points, there is at least one point between them.)

2 There is at least one point G between A and F. (For any two points, there is at least one point between them.)

3 G is between A and B. (If G is between A and F, and F is between A and B, then G is between A and B.)

4 There is one point H between A and G. (For any two points, there is at least one point between them.)

5 H is between A and B. (If H is between A and G, and G is between A and B, then H is between A and B.)

Similarly, you can show that there is a point between H and A, say M, that is also between A and B. You can show that there is a point, say J, between M and A, that is also between A and B. This locates five different points between A and B. Since you can continue indefinitely in this manner, there must be infinitely many points between A and B. Notice that the term *locate* is not used here in a physical sense. To say that you can locate 10, 20, or as many points as you wish, between two given points, means that 10, 20, or as many points as you wish to think of, exist between two given points.

Notice also that property 1 tells you that, whenever you have a situation involving betweenness, you must also have a situation involving collinearity. Property 2 tells you that whenever you have a situation involving collinearity, you must also have a situation involving betweenness. Thus, the first two properties assert that collinearity and betweenness must be considered together.

Since a line segment consists of two points and all points between these two, a line segment must be a set of collinear points. Also, since there are infinitely many points between two given points, a *line segment* is an infinite set of collinear points. Here, just as in a study of numbers, is one infinite set (a line segment) that is a subset of another infinite set (a line).

The idea of congruence is another undefined basic concept in the geometry being developed. What is meant by congruence can be expressed by saying that geometric figures are *congruent* when they are the same size and shape. Since line segments are always the same shape, only size need be considered when thinking of congruent segments.

We make a distinction between the idea of congruence and the idea of equality of line segments. Only when we have the same segment with two different names do we say that they are equal ($\overline{AB} = \overline{BA}$). Equal segments are necessarily the same size and shape; so they are congruent ($\overline{AB} \cong \overline{AB}$, or $\overline{AB} \cong \overline{BA}$). Congruent segments are not necessarily equal segments. If $\overline{AB}$ is the same size as $\overline{CD}$, then $\overline{AB} \cong \overline{CD}$. $\overline{AB}$ is a different set of points from $\overline{CD}$, and therefore we do not say that $\overline{AB}$ equals $\overline{CD}$. Two sets are equal if and only if they contain the same members.

If $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$. This relation is said to be symmetric. If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$. This relation is said to be transitive. $\overline{AB} \cong \overline{AB}$ is an example of the reflexive relation. Only the transitive property of congruence is developed in the lesson. The students will intuitively understand that congruence is also reflexive and symmetric, but these terms need not be mentioned. All of these properties of congruence will be useful in answering some of the "On your own" exercises.

Instruction notes

■ After discussing exercise F on page 37, ask your students to locate any two points on a piece of paper and represent them by dots. Say: "Now

nake a sketch of a line segment whose endpoints re represented by the two dots. Try to locate be- ween the two dots a point that is not contained n the line segment. Can this be done?" [No]

Continue with the exercises through L on age 37. Have students represent three collinear oints by dots on their papers. Tell them to label he dots P, Q, and R. Ask: "Is P between Q and ?" [Answers will vary according to the ways the lots are labeled.] "Is Q between P and R? Is R etween P and Q? Can you answer 'Yes' to more han one of these three questions?" [No]

Proceed with the developmental exercises hrough the paragraph following C on page 38. Ask members of your class to give two different names for each of three segments represented n D4.

Complete the developmental exercises. Review, with your students, the properties that are developed in this lesson.

Provision for individual differences

■ In teaching the properties of betweenness to slow learners, you might have the students demonstrate some of the properties. Each student could "represent" a point, and a group of students in a row could "represent" collinear points.

The slow learners might also make pictures illustrating each of the properties of betweenness and each of the properties of congruent segments. [If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$; $\overline{AB} \cong \overline{AB}$; if $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$.]

1 / 11 Exploring ideas / Pages 41-44

Comparing segments

Objectives

■ The students use the idea of betweenness to help them understand the idea of separation. Then they use betweenness and separation to compare segments.

Content overview

■ In lesson 10 the idea of betweenness for points was used to develop the idea of a line segment. In this lesson betweenness is used to develop the idea of separation.

The idea of *separation* is applied to lines, planes, and space. A plane separates space into two half-spaces. A line separates a plane that includes the line into two half-planes. A point separates a line that contains the point into two half-lines. A plane is the boundary of each half-space into which the plane separates space. A line is the boundary of each half-plane into which the line separates the plane. A point is the boundary of each half-line into which the point separates the line. The boundary (a plane, a line, or a point) is not included (or contained) in either of the two sets (half-spaces, half-planes, or half-lines) determined by the boundary. The two half-spaces (or half-planes or half-lines) determined by a plane (or a line or a point) are disjoint sets.

The idea of betweenness can be used to decide if two points that are not in the boundary are in different half-spaces (or half-planes or half-lines) determined by the boundary. If plane ABC contains a point that is between points D and E, then D and E are in different half-spaces determined by plane ABC. If ℓ_1 is included in plane ABC, then ℓ_1 separates plane ABC into two half-planes. If points G and H are in plane ABC but not in ℓ_1, and ℓ_1 contains a point that is between G and H, then G and H are in different half-planes determined by ℓ_1. If point F is in ℓ_1, then F separates ℓ_1 into two half-lines. If points J and K are in ℓ_1 and F is between J and K, then J and K are in different half-lines determined by point F.

The ideas of congruence, separation, and betweenness are used to develop an understanding

of what is meant by the relations "greater than" and "less than" for segments. These relations are developed without the use of measures.

Since both of the statements $\overline{AB} < \overline{CD}$ and $\overline{CD} < \overline{AB}$ cannot be true at the same time, the relation of "less than" is not symmetric. By the same reasoning, the relation of "greater than" is not symmetric.

$\overline{AB} < \overline{AB}$ and $\overline{AB} > \overline{AB}$ are both false statements. Therefore, neither relation is reflexive.

Both relations ("less than" and "greater than") are transitive, and this fact is developed for the relation of "less than" at the end of the lesson. As was done in lesson 10, the shorter way of expressing the transitive property is used. That is, instead of saying, "If $\overline{AB} < \overline{CD}$ and $\overline{CD} < \overline{EF}$ are true statements, then $\overline{AB} < \overline{EF}$ is a true statement," we say, "If $\overline{AB} < \overline{CD}$, and $\overline{CD} < \overline{EF}$, then $\overline{AB} < \overline{EF}$."

Instruction notes

■ Students will more easily understand the ideas presented in this lesson if you use visual objects in teaching the lesson. To represent a plane, you can use your chalkboard or a piece of tagboard. You can represent a line by using a dowel stick, a metre stick, a knitting needle, or a yardstick. You can represent points by using small tagboard disks, dimes, or thumbtacks. If you use a sheet of tagboard, a metre stick, and tagboard disks, you might develop the idea of a plane separating space by using the following procedure:

Place the tagboard on an easel or on the chalkboard trough. Say: "This piece of tagboard suggests a plane. Does this plane extend endlessly?" [Yes] "Now think of this plane as separating space into two half-spaces. An object in space must be in one of three sets of points: in one half-space, in the other half-space, or in the plane. Think of two points in space. Suppose we want to know if the two points are in the same half-space." Hold two disks in front of the tagboard. "How do you know that there is no point in the plane that is between the points represented by these disks?" [The two points may be joined by a line segment that does not intersect the plane.] "When there is no point in the plane between two points, can we conclude that the two points are in the same half-space?" [Yes]

Hold one disk in front of the tagboard and one disk behind the tagboard. "Explain how you know that there is a point in the plane between these two points." [A line segment joining the points must intersect the plane.] Move up the disk that is behind the tagboard so that it can be seen but is still behind the tagboard. "Is a point in the plane between these two points?" [Yes] "Explain your answer." [The plane extends beyond the piece of tagboard. When there is a point in the plane between two points, then the two points are in different half-spaces.]

Continue to move the two disks around in different positions and let your class decide if the two points are in the same half-space for each position. For one position, put a disk on the tagboard and hold another disk in front of the tagboard. Make sure your students understand that, in this position, one point is in a half-space and one point is in the boundary (the plane). The points are not in different half-spaces.

You can use similar procedures in developing some of the other ideas presented in this lesson. You can use masking tape to fasten a metre stick to the surface of the chalkboard to obtain a model that can be used in developing the idea of a line separating a plane into two half-planes. You will find objects from a set of wooden building toys useful in developing the idea of a point separating a line into two half-lines.

You may want to devote two class periods to this lesson. The developmental exercises through T on page 42 and "On your own" exercises 1 through 13 might be covered on the first day.

As your students study the ideas presented in developmental exercises A through M on pages 42 and 43, they should understand that they are using previously studied ideas about separation and congruence in deciding if one segment is less than another segment. They should also understand that they are not using measures when making comparisons of line segments in this lesson.

Provision for individual differences

■ Abler students should have no difficulty in grasping the ideas presented in this lesson. You might have these students list the ideas of separation presented and test each idea by using other geometric objects. For example, "a plane is separated by a line" could be tested by considering whether or not a line segment separates a plane, and so on.

Slow learners may find it difficult to visualize the ideas presented in the lesson. For these students, do not omit any of the developmental exercises unless you demonstrate the ideas for the class. Always try to use as many visual aids as possible when developing geometric ideas with slow learners. You may find it profitable to give close attention to how these students answer "On your own" exercises 1 through 13. Utilize sketches as exercises 20 through 29 are discussed in class.

1 / 12 Exploring ideas / Pages 45-49

The circle

Objectives

■ The students learn that the solution set of a geometric condition is a geometric figure. They study the circle, the interior of a circle, and the exterior of a circle.

Content overview

■ Many students who have studied a traditional mathematics program have an intuitive idea of what a circle is, but they have not learned how a circle is defined mathematically. This is because they do not know about geometric conditions and the relations of congruence, of "less than," and of "greater than." In the past, students often thought of the interior of a circle as being part of the circle.

The expressions "condition," "universe," "solution," "solution set," and "satisfy a condition" have the same meaning for sets of points as for sets of numbers. A circle is a set of points that is the solution set for a geometric condition such as $\overline{MX} \cong \overline{MN}$ when the universe for the variable X is a plane. Even though the condition involves segments, the solutions are points. Each member of the solution set of the condition $\overline{MX} \cong \overline{MN}$ is a point in the universe that, together with M (the center), determines a segment congruent to $\overline{MN}$. Since point N satisfies the condition $\overline{MX} \cong \overline{MN}$, it is a solution.

$\{X \mid \overline{MX} \cong \overline{MN}\}$ is a standard description of circle M. The interior of circle M is $\{X \mid \overline{MX} < \overline{MN}\}$; the exterior of circle M is $\{X \mid \overline{MX} > \overline{MN}\}$. Although the following notation is rarely used, the interior of circle M is also $\{X \mid \overline{MN} > \overline{MX}\}$, and the exterior is also $\{X \mid \overline{MN} < \overline{MX}\}$. The universe for X is plane MNR.

Note that each point in the universe satisfies just one of the conditions $\overline{AX} \cong \overline{AB}$, $\overline{AX} < \overline{AB}$, or $\overline{AX} > \overline{AB}$. Note that the centre of a circle and the points in a radius, a chord, and a diameter of a circle are all coplanar points. Also note that, for a radius, the endpoint that is not the centre is in the circle, and that, for a chord and a diameter, the two endpoints are in the circle. The other points in a radius, a chord, and a diameter of a circle are in the interior of the circle.

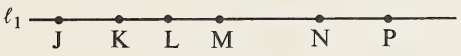
More extensive work with plane figures and their regions is provided in unit 2.

Instruction notes

■ You can develop the ideas in this lesson by using the developmental exercises, or you can use the following procedure in a class discussion.

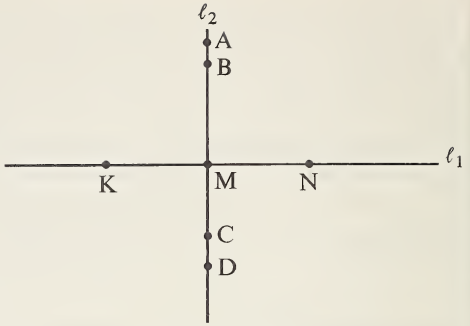
Say: "You learned that you could express the fact that two segments are congruent by writing a sentence such as $\overline{AB} \cong \overline{CD}$." Write "$\overline{AB} \cong \overline{CD}$" on the chalkboard. "Notice that this sentence does not express a condition. It does not contain a variable." Write "$\overline{MX} \cong \overline{MN}$." "The X in this condition is a variable. Does this sentence express a condition?" [Yes] "What kind of universe will the variable X have?" [A set of points]

Write "$\overline{MK} \cong \overline{MN}$" on the chalkboard, and make the following sketch:



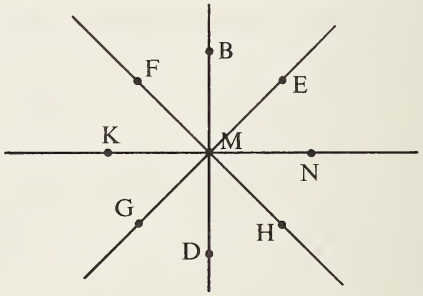
"Suppose that the universe for $\overline{MX} \cong \overline{MN}$ is ℓ_1. Does point N satisfy the condition?" [Yes] "Explain why point J does not satisfy the geometric condition $\overline{MX} \cong \overline{MN}$." [$\overline{MJ} > \overline{MN}$ because point L is between point M and point K. Therefore, J cannot satisfy the condition.] In a similar manner, have students explain why points L and P do not satisfy the condition. "Segment MK is congruent to segment MN. Explain how you know that point K satisfies the condition." Erase the dots for points K and N and make new dots with colored chalk for these points that satisfy the condition. Say: "I have used colored chalk to represent the points that satisfy the condition."

Write the sentences $\overline{MB} \cong \overline{MN}$ and $\overline{MD} \cong \overline{MN}$ on the chalkboard. "Now let us see what points satisfy the geometric condition $\overline{MX} \cong \overline{MN}$ when the universe for X is ℓ_1 and ℓ_2." Sketch the figure represented in the next column.

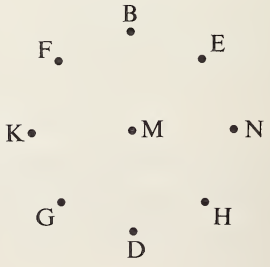


"Which points in ℓ_2 satisfy $\overline{MX} \cong \overline{MN}$?" [B and D] Erase the dots for B and D and make new colored dots in your sketch, as you did for points K and N.

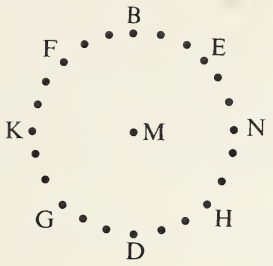
Continue with this procedure, using four lines for the universe, as shown below.



Each point indicated by a dot, except point M, is a solution of $\overline{MX} \cong \overline{MN}$. Now erase the pictures of the lines, so that the sketch looks like this:



Suggest that the universe for X is the plane suggested by the chalkboard. The students should understand that there are many points in the plane that satisfy the condition. Make dots for more of the points, as shown below.



Then say: "We can sketch a simple closed curve that will indicate all the points that satisfy the condition." Sketch the curve. Ask: "What geometric figure is represented by this picture when we think of the universe as a plane?" [A circle]

Represent the circle by using colored chalk in your sketch.

Introduce the terms *circle*, *centre*, *radius*, *chord*, and *diameter*. Ask your students to tell in what ways radii, chords, and diameters are alike and in what ways they are different.

You may use a procedure similar to the one outlined above to teach the notions of the interior and exterior of a circle. For its visual effect, use chalk of different colors to show the circle, its interior, and its exterior.

Provision for individual differences

■ Many concepts new to the students are developed in this lesson. Therefore, do not omit any developmental exercises unless you have developed all the ideas without using the text.

Ask your abler students to consider the solution set of $\overline{MX} < \overline{MN}$ and $\overline{MX} > \overline{MN}$ when the universe for X is space. [Interior and exterior of a sphere] Ask them also to consider the solution set of $\overline{AX} \cong \overline{CD}$ when the universe is a plane that does not contain point A. [A circle, or the empty set, or a point]

Have the slower students reinforce their control of the vocabulary associated with the circle. Utilize procedures that require their participation. For example, as one student at the board traces the diameter of a circle, have one of the students who is seated give the name "diameter" orally. If possible, have on hand a model of a circle. This can be passed around the room. Have the various students show, by tracing with their fingers, where the chord, diameter, radius, and so on, are located on the model.

1 / 13 Exploring ideas / Pages 50-54

Number lines and graphs

Objectives

■ The students are introduced to the notion of a number line. They learn to use a picture to represent a number line and to make a graph of a set of natural numbers.

Content overview

■ In this lesson we develop the idea that a natural-number line involves an association of points in a line with natural numbers. This association involves the assignment of a unique point to a natural number and a unique natural number to a point in a line.

A natural-number line is an abstraction, but it can be represented by symbols. The notion of a natural-number line can be represented as follows: by a picture of a line, dots to represent points that are associated with natural numbers, and numerals for these numbers. Selected points of a line, separated by congruent segments, are

associated with consecutive natural numbers. It is customary, but not necessary, to place the dot representing point 1 to the right of the dot representing point 0. The segment determined by these two points is chosen arbitrarily and is called the "unit segment." Each segment determined by a pair of points associated with consecutive natural numbers is congruent to the unit segment. In drawing a picture of a number line, congruent segments ideally should be constructed by using compasses, not by using a graduated ruler.

In this lesson the lines are pictured horizontally. This is an arbitrary, though customary, choice.

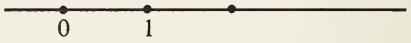
A *graph* is defined in the lesson as a set of dots. In some books, a graph is thought of as a set of points, but in our development we think of points as ideas that are represented by dots. It is only when you think of a graph as consisting of dots that it makes sense to use phrases such as "look at the graph" or "make a graph." From the definition, it follows that a picture of a natural-number line is an incomplete graph of the set of natural numbers. The term "incomplete" is used because not all the dots whose points are associated with natural numbers can be shown. To avoid confusion, the word "graph" is used only as a noun. We say "make a graph of the set" instead of "graph the set."

A graph may indicate a finite or an infinite set of numbers. A graph of a solution set is included in a graph of the universe. To make a graph of a solution set, the student should first make a graph of the universe and then indicate which dots are in the graph of the solution set.

Instruction notes

■ You can develop the ideas presented in the lesson in the following manner.

Read the introductory paragraph with your class. Have students sketch a picture of a horizontal line segment, placing two dots near the left endpoint. Say: "You are going to associate the number zero with the point represented by the dot farther to the left and the number one with the point represented by the other dot. Write the numerals 0 and 1 under these dots. Do the two points represented by dots in your picture determine a line segment?" [Yes] "Using a dot, represent another point in the line to the right of point 1 so that point 1 and this new point determine a segment that is congruent to the segment determined by points 0 and 1." The sketch should now look like the picture below.



Walk around the classroom to check each student's work. Then ask: "What number should you associate with the last point you located?" [2] "How will you locate the point that you will associate with the number three?" [Locate it to the right of point 2 so that points 2 and 3 determine a segment congruent to the segment determined by points 0 and 1.]

"Was it necessary for each of you to use the same size segment between points 0 and 1?" [No] "When you associate numbers and points, does the first segment you locate have to be a certain size?" [No]

Continue in this manner until you have presented all the ideas developed in the lesson. Then have the students work the "On your own" exercises. If they have difficulty with any of them, they should carefully study the developmental exercises.

To provide further practice, use your chalkboard to show how both finite and infinite universes can be indicated on pictures of natural-number lines, and how the solutions of conditions can be indicated by encircled dots. Use a condition for which the solution set is the universe

[$x + 1 > 0$. U = N.] and a condition for which the solution set is the empty set. [$x < 4$. U = {4, 5, 6, 7, 8, 9}.]

You can make your own transparencies and use an overhead projector to demonstrate how graphs are made.

Provision for individual differences

■ The ideas developed in this lesson will not present any difficulty for abler students. You can have them study the developmental exercises independently, and then follow this with general class discussion before having them work the "On your own" exercises.

For slower learners, be sure to emphasize the following points:

1 In a number line we are matching natural numbers with points in a line.

2 We can use a picture of a number line to draw a graph of a set of numbers.

3 We can use a graph of a set of numbers to make a tabulation of that set of numbers.

Give slow learners individual help as they draw their first graphs. Have students from the class make graphs on the chalkboard for the first few "On your own" exercises. Be sure that they understand how to draw the graph of an infinite set of numbers before you assign exercises 14 through 23.

---

1 / Checking up / Page 54

Inventory tests

■ These tests are the only inventory tests included in *STM 1*.

These tests are designed to help you find out how well the students can compute when using natural numbers. Therefore, the tests should be used mainly for diagnostic purposes. The tests can be used to identify the students who need help with their computation.

The three lessons that immediately follow the inventory tests concern multiplication and division computation involving natural numbers. The students' performance on the inventory tests will help you decide how much emphasis should be placed on these three lessons.

If the students show average success on the tests, you might devote one period to each of the three lessons. If most of the students do well, you can cover all three lessons in one period and give individual help to those who need it.

---

1 / 14 Learning computation / Pages 55-56

Multiplication

Objectives

■ Students take another look at multiplication involving natural numbers.

Content overview

■ This lesson is concerned with the understandings and skills involved in finding the product of two natural numbers. The method of computation presented in this lesson is similar to that presented in the *STA* program.

The term *multiplicand* refers to the number multiplied. The term *multiplier* refers to the number by which the multiplicand is multiplied. When 2 is multiplied by 3, the multiplicand is 2, the multiplier is 3, and the number obtained is the *product*.

In unit 5 we develop the idea that the numbers being multiplied are factors of the product. 2 and 3 are both considered factors of 3×2, or 6.

The algorism for the multiplication of natural numbers involves the distributive property. This

property is developed on pages 274 through 277 of the student's book. Consider the example 8×34. We can express this as $8(30 + 4)$, and by applying the distributive property, $8(30 + 4)$ means $8(30) + 8(4)$. This property can be stated either as $a(b + c) = ab + ac$ or as $(b + c)a = ba + ca$. Consider the example 26×416. Think of the multiplier, 26, as $(20 + 6)$. By using the idea of the distributive property, $(20 + 6)416 = 20(416) + 6(416)$. Since the multiplication algorism presented in this lesson is based on the distributive property, the student using this algorism should understand that, when the multiplier is a number such as 26, the multiplicand is multiplied by 20 and by 6.

Remember that the terms multiplicand, multiplier, and product refer to numbers, not to numerals. Thus, a multiplicand is not a "three-digit numeral." It may be a number that is expressed by a three-digit numeral, however.

Instruction notes

■ You may want to give your students an inventory test to determine how much time you should devote to learning computation in lessons 14, 15, and 16. You can use the tests on page 54 of the student's book for this purpose.

You can provide your students with practice in using the terms presented in this lesson by writing examples such as "$9 \times 42 = 378$" on the chalkboard and asking them to identify the multiplicand, the multiplier, and the product. Then rewrite the examples in the vertical form and repeat the identifying process.

If you have your students follow the developmental exercises in the student's book, they should understand that the red numerals express the numbers they are to think about as they study each example.

Write the multiplicand and the multiplier for exercises A through F on page 56 on the chalkboard. Tell the students to complete these computations at their desks and check their results with the work shown in the text. Give individual help to students who need assistance.

Provision for individual differences

■ Abler students may be interested in experimenting with different ways of multiplying numbers. Give them an example such as 598×736 to multiply in the way shown at the right. Have them test other examples so that they can learn to decide how many partial products there will be when computing with this method. [The number of partial products can be found by multiplying the number of digits in the numeral for the multiplicand by the number of digits in the numeral for the multiplier.]

67	
104	
28	(4×7)
240	(4×60)
0	(0×7)
0	(0×60)
700	(100×7)
6000	(100×60)
6968	

For a class that has a particular interest in multiplication, you may wish to demonstrate that if the tens in a multiplier expressed by a two-digit numeral are multiplied first and the ones second, the product will be the same as the product obtained by multiplying in the conventional manner. Suggest that the students try multiplying in a different order when the multiplier is expressed by a three-digit numeral (that is, multiplying the tens, hundreds, and ones, in that order).

You may find that some of your students do not understand how to use the multiplication algorism or that they are very weak in their knowledge of the basic facts. Give as much individual help as you can to these students. You may use pages 9 through 26 in *Seeing Through Arithmetic, Special Book B** to reteach multiplication.

---

* Maurice L. Hartung, Henry Van Engen, and Lois Knowles, *Seeing Through Arithmetic, Special Book B* (Toronto: W. J. Gage Limited, 1962).

1 / 15 Learning computation / Pages 56-58

Preparation for division

Objectives

■ The students learn to round a number to the next greater or the next lesser multiple of a power of ten as preparation for estimating quotients in division of natural numbers.

Content overview

■ It is sometimes helpful to round numbers when using the division algorism that is developed in lesson 16. If you are not familiar with this algorism, which is developed in the *STA* program, read lesson 16 in *STM 1* and the lesson notes for lesson 16. When using this algorism, computation may, at times, be greatly simplified if divisors are rounded up and dividends are rounded down. A partial quotient obtained by this method is never too great.

Suppose that we are to divide 4294 by 68. We round 68 up to 70 and round 4294 down to 4200. The computation is shown below. Notice that, while we use the rounded numbers to obtain partial quotients, we use 4294 and 68 in the computation.

(a) 4200 ÷ 70 is the same as the quotient 420 ÷ 7, or 60.
(b) 60 × 68 = 4080.
(c) 4294 − 4080 = 214.
(d) Round 68 up to 70 and round 214 down to 210.

```
68)4294 | 60
   4080
   ----
    214 |  3
    204
    ---
     10 | 63
```

(e) 210 ÷ 70 is the same as the quotient 21 ÷ 7, or 3.
(f) 3 × 68 = 204.
(g) 214 − 204 = 10.
(h) Since 10 is less than 68, we cannot remove another group of 38.
(i) The quotient is 60 + 3, or 63. The remainder is 10.

Because decimal numerals are used to express numbers, we round a number to the next greater or the next lesser multiple of a power of ten. The positive integral powers of ten are 1, 10, 100, 1000, and so on. It is important to remember that, when a number is rounded to the next greater ten, it is rounded to the next greater multiple of ten. Thus, if 82 is rounded to the next greater ten, it is rounded to 90, not 92. Similarly, when the number 592 is rounded to the next lesser hundred, it is rounded to 500, not 502 or any other number. Students who have not been taught to round numbers in earlier grades may make the errors discussed in this paragraph and, consequently, may have difficulty in using the division algorism presented in lesson 16.

Attention is given to the terms *divisor*, *dividend*, and *quotient*. These terms are frequently used in lessons 15 and 16.

In working with partial quotients, the students must understand that the quotient remains unchanged when the divisor and the dividend are each divided by the same number. At the end of the lesson, the idea is presented that a dividend and a divisor can each be multiplied by the same number without affecting the quotient. This idea is given more emphasis and attention when division using numbers expressed by decimal fraction numerals is discussed.

Instruction notes

■ If the results of inventory tests 9 and 10 on page 54 indicate that the students have a good understanding in using a division algorism and that they possess adequate skills in computation, devote little class time to lessons 15 and 16. Students who have mastered one algorism may resist learning a new procedure. If this is the case, do not insist that they adopt a new procedure.

For students who do not know how to divide, devote enough time to enable them to achieve an understanding of the algorism presented in this book. Slow learners can be helped in understanding how to divide with the algorism presented in lesson 16.

To provide practice in using terms, write sentences such as "$72 \div 8 = 9$" and "$108 \div 12 = 9$" on the chalkboard. For each sentence, have a student identify the numerals that express the dividend, the divisor, and the quotient.

Provision for individual differences

■ Slow learners may need more practice in rounding numbers. Try to diagnose the causes of each student's difficulty and give individual instruction, if possible.

You may tell your abler students to try to give rules for finding natural numbers that the dividend and the divisor can be divided by so that a remainder of 0 is obtained. If a remainder of 0 is obtained when a given natural number is divided by a second natural number, then the given natural number is said to be divisible by the second natural number.

You might make this a discovery or "research" activity. The rules listed below are examples of what the students might find:

1 If the last digit in the numeral for the dividend and for the divisor is 0, 2, 4, 6, or 8, both the dividend and the divisor are divisible by 2.

2 If the sum of the numbers expressed by the digits in the numeral for the dividend and the sum of the numbers expressed by the digits in the numeral for the divisor are both divisible by 3, then both the dividend and the divisor are divisible by 3.

3 A number is divisible by 4 when the number expressed by the last two digits of the numeral is divisible by 4.

4 A number whose numeral ends in two zeros is divisible by 4.

5 A number is divisible by 5 if the last digit in the numeral for the number is a 0 or a 5.

6 A number is divisible by 8 if the number expressed by the last three digits of its numeral is divisible by 8. (A number whose numeral ends in three zeros is also divisible by 8.)

7 A number is divisible by 9 if the sum of the numbers expressed by the digits in the numeral for the number is divisible by 9.

---

1 / 16 Learning computation / Pages 58-60

Division

Objectives

■ The students learn to relate the process of division involving natural numbers to that of repeated subtraction.

Content overview

■ The division algorism introduced in this lesson is presented in the *STA* program.

Tests 9 and 10 on page 54 may be used as inventory tests. Students who have no difficulty with division may use lesson 16 for practice material.

Division is generally regarded as the most difficult basic process in arithmetic. One reason for this is that division is not a direct process like addition and multiplication. Division involves some estimating and adjusting in all but the simplest examples. A second reason for confusion among elementary pupils is that often two different methods for division are employed. "Short division" is first introduced, followed by "long division." The two methods are sufficiently different to create difficulties for pupils who cannot relate the two methods. A third reason is that the computational form (or algorism) for "long" division commonly used in this country is difficult

o understand. In teaching this traditional computational form, it is difficult for teachers to relate the division process to that of repeated subtraction.

In the past, the conventional way of writing a division example in this country was to place the numeral for the quotient above the numeral for the dividend. This scheme has only been used for the past thirty or forty years, however, and it has never been popular in other scientifically advanced countries.

The algorism for division presented in this lesson is based on the idea that subtractions of the divisor may be repeated until the dividend is "used up." The number of possible subtractions is the quotient.

Repeated subtraction of the divisor can result in a rather tedious process, however, and a refinement can be introduced to reduce the number of subtractions required. For example, to divide 2585 by 55, using the idea of repeating the subtraction of 55, would require forty-seven separate subtractions. The first three examples shown below illustrate an order of increasing maturity in estimating partial quotients. Notice that the computation in the third example is greatly simplified as compared to the first example. The fourth example shows the division using the traditional algorism.

											47
55)2585	10	55)2585	30	55)2585	40	55)2585					
550		1650		2200		220					
2035	20	935	10	385	6	385					
1100		550		330		385					
935	10	385	6	55	1						
550		330		55							
385	5	55	1		47						
275		55									
110	2		47								
110											
	47										

Partial quotients that are determined by rounding the dividend to a lesser number and rounding the divisor to a greater number are never too great. For the third example, 55 is rounded to 60 and 2585 is rounded to 2400. 2400 is used rather than 2500 because 2400 can be more easily divided by 60. Since the quotient $2400 \div 60$ is the same as the quotient $240 \div 6$ (or 40), 40 is the first partial quotient. If 40 groups of 55 are subtracted from 2585, then 385 is left. 385 is then rounded to 360. $360 \div 60$ has the same quotient as $36 \div 6$, or 6. If 6 groups of 55 are subtracted from 385, then 55 is left. Since 55 contains 1 group of 55, the quotient of 2585 and 55 is $40 + 6 + 1$, or 47.

The method of estimating partial quotients involves these steps:

1 Round the divisor up to the next greater ten, hundred, and so on, so that the first digit is the only non-zero digit. Round the dividend down, so that the rounded dividend is easily divisible by the rounded divisor. The partial quotient should be a number of tens, hundreds, etc. To divide 2963 by 84, round 84 to 90, and round 2963 to 2700. The first partial quotient is 30.

2 Divide both rounded divisor and rounded dividend by a multiple of 10. Thus, in the example above, think of $2700 \div 90$ as $270 \div 9$.

Notice the "Keeping skillful" exercises on page 60. There are forty-eight such blocks of exercises in *STM 1* that you can use to ensure adequate review of ideas presented in earlier lessons.

Instruction notes

■ In teaching this lesson you can discuss the examples as they are developed and have students follow the presentation in their books; or you can discuss each example with the students and have them perform the necessary computation.

If students have learned to use another division algorism and are proficient in using that compu-

tational process, do not insist that they use the method shown in this lesson. A discussion of this lesson may help them better understand the algorism they use, but you may gain little and lose a great deal if you insist that they forget one method and master another.

Provision for individual differences

■ Abler students might consider the following questions and give two examples that substantiate each of their answers:

1 The dividend is multiplied by 2; the divisor is unchanged. How does the quotient compare with the original quotient? [It is twice as great.]

2 The dividend is multiplied by 4; the divisor is multiplied by 2. How does the quotient compare with the original quotient? [It is twice as great.]

3 The dividend is multiplied by 10; the quotient is multiplied by 5. By what number should you multiply the original divisor to obtain a true statement? [2]

4 The dividend is increased; the divisor is unchanged. How does the quotient compare with the original quotient? [It is greater.]

5 The divisor is decreased; the dividend is unchanged. How does the quotient compare with the original quotient? [It is greater.]

Give slow learners as much encouragement and individual attention as you can. Remember that it is important for them to understand the division algorism and divide accurately. It is not important that they work quickly or obtain a minimum number of partial quotients.

---

1 / Keeping skillful / Page 60

■ The purpose of the "Keeping skillful" exercises is to provide help in maintenance of mathematical ideas and basic skills.

This set of "Keeping skillful" exercises is concerned with computation involving the natural numbers and with finding and graphing solution sets of conditions expressed as mathematical sentences.

Since these exercises concern only material that has been developed in previous lessons, most students will experience little difficulty with them. For average and abler students, you may wish to assign all or part of a set of "Keeping skillful" exercises as work to be done outside of class. This work can be followed by class discussion on those exercises that caused difficulty.

"Keeping skillful" exercises can be especially valuable in helping slow learners master fundamental ideas and gain greater skill in computation. These students will require a great deal of time to complete all of a set of exercises. It may be advisable to have slow learners work some of the more difficult exercises in class so that you can give them individual help.

Because these comments apply, in general, to all the sets of "Keeping skillful" exercises, suggestions for each "Keeping skillful" are not included in this *Teaching Guide*.

---

1 / 17 Exploring problems / Pages 61-64

Problems involving conditions for equality

Objectives

■ The students learn to write sentences that express conditions for equality for additive and subtractive problem situations.

Content overview

■ Problems are more easily solved if a systematic approach is used in finding the solution. A problem should first be analyzed for pertinent information. A mathematical sentence can then be

written to show the structure of the problem. This sentence expresses a condition for the problem. It remains simply to find the solution set of the condition and to use members of the solution set to find an answer to the problem. This approach is consistent with the approach used in grade 9.

Only additive and subtractive situations (with real or imagined action) that involve conditions for equality are used in this lesson. Problems that involve conditions for inequality are considered in lesson 18.

In this lesson, the emphasis is on developing conditions for problems. Methods by which solutions for conditions and answers to problems are found are developed in lessons 19, 20, and 21.

For the problem in D1, a group of n quarters is taken from the whole group of 15 quarters, and a group of 9 quarters is left. In problems like this one, in which a subset of a given set is removed from the given set, the action is subtractive. The pictures related to the problem in D1 should help the student to visualize the subtractive action.

The condition $15 - n = 9$ means that, when n is replaced by the number to be found, $15 - n$ will be the natural number 9. Thus, $15 - n$ and 9 each refers to the number in the group that remains.

The general form of a sentence that expresses a condition for a subtractive problem is $a - b = c$; a is a variable for the number in the whole group; b is a variable for the number in the group removed; and c is a variable for the number in the group remaining. A subtractive problem may involve finding the number in the whole group, the number in the group removed, or the number in the group remaining.

The form $a - b = c$ should not be presented to the students. We use the general form here to present the teacher with a general discussion rather than a particular one.

For the problem in D2, a group of 5 coins is put with a group of x coins and a new group of $x + 5$, or 39, coins is formed. In problems like this one, in which the members of one set are put with the members of another set to form a third set, the action is additive. The pictures related to the problem in D2 should help the students visualize the action in the problem. The condition $x + 5 = 39$ means that, when x is replaced by the number to be found, $x + 5$ will be the natural number 39. Thus, $x + 5$ and 39 each refers to the new group that is formed as the result of the additive action.

The general form of a sentence that expresses a condition for an additive problem is $a + b = c$; a is a variable for the number in the original group; b is a variable for the number in the group that is put with the original group; and c is a variable for the number in the new group that is formed or imagined to be formed. An additive problem may involve finding the number in the original group, the number in the group that is put with the original group, or the number in the new group that is formed.

The conditions $15 = 9 + n$, $15 = n + 9$, $9 - 3 = n$, $n = 6$, and so on, are equivalent to the condition $15 - n = 9$ since, for a given universe, they have the same solution set as $15 - n = 9$. These conditions, however, do not reflect the action in the problem in D1. They do not represent the problem situation as it is given. We think that students should learn to translate the problem situation as it is given by writing an open sentence that expresses a condition for the problem. Similarly, the condition for the problem in D2 should be expressed as $x + 5 = 39$ rather than as $5 + x = 39$ or any other equivalent condition. $x + 5$ means that 5 objects are joined with x objects, but $5 + x$ means that x objects are joined with 5 objects. If the order in which objects are

joined is clear in an additive problem situation, this order should be reflected in a sentence that expresses a condition for the problem.

Comparison problems involving subtraction are also considered in this lesson. In previous problems involving subtractive action, part of a group was removed from the whole group. In comparison problems involving subtraction, two groups (often of different kinds of objects) are compared. We can imagine that part of the larger of the two groups is removed. The part that is mentally removed has the same number of members as the smaller of the two groups that are being compared. For the problem in D7, a group of 37 dimes is compared with a group of 26 nickels. The imagined action consists of removing 26 dimes from the group of 37 dimes. The number of dimes in the group remaining is the difference between the number of dimes and the number of nickels.

The general form of a sentence that expresses a condition for a comparison-subtraction problem is $a - b = c$; a is a variable for the number of objects in the larger group; b is a variable for the number associated with the subset of the larger group that is equivalent to the smaller group; c is a variable for the difference.

Instruction notes

■ Ask your students to recall what they have learned concerning conditions, universes, and solution sets of conditions in previous lessons. Tell them that in the next four lessons they will use these ideas to find answers to problems. In this lesson they will learn to analyze certain kinds of problems and to write sentences expressing conditions for the problems, but they will not find solutions of the conditions or answers to the problems. This lesson is the first of a series of steps in a systematic approach to problem solving.

You can use the examples in this lesson as they are developed. After discussing the section that involves $x + 5$ in the second column on page 61, say: "You have found that you can write '$x + 5$' to indicate that 5 coins were combined with x coins. What would $5 + x$ indicate for this problem?" [That x coins were combined with 5 coins] "Explain why you should use $x + 5$ and not $5 + x$ to express what happened in the problem." [$x + 5$ indicates that 5 coins were combined with x coins. $5 + x$ would indicate that x coins were combined with 5 coins.]

Have the students read the problems in D3 and D4 on page 62. Then, for each problem, let them try to write a sentence that expresses a condition. If your class can do this successfully, you can omit discussing exercises F through N.

After completing exercise H on page 63, say: "The comparison problems that are used in this lesson are the 'how many more' or 'how many fewer' type. To find the answer to 'how many more' or 'how many fewer,' the number associated with a subset of a group can be subtracted from the number associated with the larger group. The difference tells us how many more or how many fewer objects there are in one group than in another. A condition for a comparison-subtraction problem of this type can be expressed by a sentence such as $8 - 5 = x$. But when you compare a group of one kind of object with a group of another kind of object, can you remove the smaller group from the larger group?" [No] "Explain how we can compare two groups, each having different objects." [You think of a number of objects of the larger group equal to the number of objects in the smaller group and subtract this number from the number in the larger group.]

Continue with the developmental exercises through exercise N on page 64.

Provision for individual differences

■ If your students have been using the *STA* program, they will find lessons 17 and 18 rather easy. Abler students in particular should be able to

ead the developmental exercises and complete he "On your own" exercises in these two lessons vith little or no help. For students who have not vorked with the *STA* program, utilize these les- ons to provide a sound framework for an ap- proach to problem solving.

Slow learners need considerable help and prac- ice to visualize the action in problems. Have hese students describe the action in some of the 'On your own" problems before they write sen- ences for the conditions. For problem 3, they might say: "Judy has to make 12 posters, and she has already made 8 of these posters. n is a variable for the number of posters she needs to make. The 8 posters she now has and n posters are 12 posters. We can write the sentence $8 + n = 12$ to express a condition for the prob- lem." These students could use markers to show the action in the problems.

1 / 18 Exploring problems / Pages 65-68

Problems involving conditions for inequality

Objectives

■ The students learn to write sentences that express conditions for inequality for additive and subtractive problem situations.

Content overview

■ This lesson, which is similar to lesson 17, is concerned with only one step in the formal problem-solving process, writing a sentence that expresses a condition for the problem. However, in this lesson students work only with problems that involve conditions for inequality.

The developmental exercises in this lesson are designed to help students mentally view the action and sequence in a problem. In all problem solving, the conditions should reflect the real or imagined action.

Notice that, with this problem-solving approach, there is no need for students to seek key words as clues to the kind of action found in a particular problem.

As in the previous lesson, the students do not find the solutions of the conditions or the answers to the problems.

Instruction notes

■ As you introduce this lesson, tell your students that they will learn about another kind of problem. For these problems they will write sentences that express conditions for inequality. Explain to them that because this part of the problem-solving process is very important and must be correctly done, they are to concentrate only on learning to write the sentences, and they need not at this time be concerned about finding solutions of conditions or answers to the problems.

In teaching the lesson, you can present the idea of "imagined action" in the following way: After discussing the developmental examples in the first column on page 65, ask: "Do the sentences in D1 describe any action that has actually taken place?" [No] "Do you have to imagine that action takes place in this problem?" [Yes] "Describe an imagined action suggested by the problem." [You could imagine that Allan's airplanes are put with Pete's airplanes.] "What sum can you use to indicate this action?" [$10 + 13$]

When exercise C on page 66 has been discussed, let your students try to develop conditions for the problems in D3 and D4 without using the developmental exercises. If they are successful, omit exercises D through K on page 66.

For the problem in D5, ask: "Are the solutions of $55 < n$ the same as the solutions of $73 - 18 < n$?" [Yes] "Does the sentence $55 < n$ show the meaning of this problem?" [No]

Provision for individual differences

■ For a class of abler students, you could write on the chalkboard sentences that express conditions for inequality and then have them suggest problems for each condition.

For slow learners, especially students who have not studied the *STA* program, discuss some of the "On your own" problems before making a class assignment. As you discuss the problems, have the students use markers to illustrate the action. Have them explain what kind of action is indicated; let them tell what the variable in the condition is; and have them explain why they use the idea of "less than" or of "greater than" in the condition.

Here is a supplemental list of problems that you might use as a test or as practice exercises in writing sentences that express conditions for equality and conditions for inequality. The best sentences are given after each problem. U = N.

1 Phil earned \$4 by raking lawns and \$6 by washing cars. How much money did he earn?

$$4 + 6 = n \text{ or } 6 + 4 = n.$$

2 Jack said, "If you add 7 to the number of pounds I weigh, the result is more than 110." How many pounds could Jack weigh?

$$n + 7 > 110.$$

3 Bill has more than 315 stamps in his collection. 42 of them are Canadian stamps, and the rest are European stamps. How many European stamps can Bill have?

$$42 + n > 315 \text{ or } n + 42 > 315.$$

4 Sue spent less than \$15 for a dress and a pair of shoes. If she spent \$9 for the dress, how many dollars could she have spent for the shoes?

$$9 + n < 15 \text{ or } n + 9 < 15.$$

5 Bob weighed 117 pounds before he went to camp. When he came home, he weighed 4 pounds less. How many pounds did Bob weigh when he came home?

$$117 - n = 4 \text{ or } 117 - 4 = n.$$

6 Larry mowed 18 lawns during July and 24 lawns during August. How many lawns did he mow during these two months?

$$18 + 24 = n.$$

7 In 1908 the Royal Canadian Mounted Police celebrated their thirty-fifth anniversary. In what year was this police force established?

$$n + 35 = 1908 \text{ or } 1908 - 35 = n.$$

8 Helen spent \$6 of the money she had saved. She told Ann, "The number of dollars I spent is the same as the number of dollars I have left." How many dollars had Helen saved?

$$n - 6 = 6.$$

9 A British scientist predicted that a comet seen in the year 1682 would again be visible in 1758. His prediction was correct. In how many years had this comet traveled the length of its orbit?

$$1682 + n = 1758 \text{ or } 1758 - 1682 = n.$$

10 In 1963, Mr. Fern bought a new car. He traded in his old car, which he had bought new fewer than 8 years earlier. In what years could Mr. Fern have bought his old car?

$$1963 - n < 8.$$

11 There were two candidates in the last election for student union president at Manning Heights Junior High School. Out of the 261 votes cast, candidate A won 83 votes. How many votes did candidate B receive?

$$261 - 83 = n.$$

12 The height of Clyde's house is 27 feet. The Bankers' Building in Clyde's town is more than 17 feet higher than his house. What can be the height of the Bankers' Building?

$$n - 27 > 17.$$

13 The fire station in Southport is 58 years old. The Southport School is older than the fire station. The difference in their ages is fewer than 20 years. How many years old can the Southport School be?

$$n - 58 < 20.$$

/ Keeping skillful / Page 68

These "Keeping skillful" exercises are concerned with geometry. These are the first exercises in *STM 1* in which the students are asked to make sketches of geometric figures and then use the sketches in answering other exercises. A demonstration of how such a sketch should be made would be helpful. When making a demonstration, emphasize the importance of following directions in the proper order.

See page 74 of this *Teaching Guide* for suggestions for using the "Keeping skillful" exercises.

1 / 19 Exploring problems / Pages 68-72

Solution sets of conditions for equality

Objectives

The students learn to solve problems by finding the solution sets of conditions for equality.

Content overview

In our problem-solving approach, the student first writes a sentence that expresses a condition for the problem. Next, he computes to find the solution set of the condition. Then he uses the solution set to determine an answer for the problem. To do this, the student must recognize the structural nature of the problem, understand how to use symbols to write a sentence expressing the condition, and determine what process can be used to find the solution set of the condition.

The problems in D1 pose the question "How many more are needed?" A general form of the sentence that expresses the condition for this problem is $a + n = c$. (Do not present this general form to students.) For the problems in D1, n is a variable for the number of place cards Mary should make and for the number of pictures of old automobiles that Jim bought.

An analysis of problem A in D1 reveals that Mary has 15 place cards to which she must add an undisclosed number of place cards to have the 27 place cards she needs. The students must first seek to recognize the structure of the problem, which in this case is additive. They next write a sentence that expresses the condition, $15 + n = 27$. Then they find the solution set of the condition, $\{12\}$. The last step in the problem-solving procedure is to give the answer to the question posed by the problem: "Mary should make 12 more place cards."

From problem B in D1, the students learn that a sentence such as $15 + n = 27$ may express a condition for many different problems.

The students are not as yet prepared to use formally the axioms of algebra and equivalent conditions to find solution sets of conditions. They will learn these in *STM 2*. Therefore, the students need to discover another, less formal, way to find the solution set of a condition. One way to find the replacement for n that makes a true statement from the condition is shown in the developmental examples. The 27 markers in the pictures on page 68 represent the 27 place cards Mary needs. 15 of the markers are dimmed in the third picture to indicate to the students that they can imagine that the 15 cards Mary already has are removed from the 27 she needs. This suggests subtracting 15 from 27, which gives the solution set of the condition. The students then use the solution of the condition to get the answer to the question in the problem.

The problems presented in D2 pose the question "How many were there to begin with when

the number added and the resulting total are known?" For problem C, the students first see that Bob had some nails to begin with, but they do not know how many. After he had found 8 more nails, he had 17. The condition $x + 8 = 17$ reflects the structure of this problem. A general form of the sentence that expresses the condition for this problem is $n + b = c$. The replacements for the variables b and c are given in the problem. n is a variable for the number of objects there were to begin with.

To find the solution set of the condition expressed by $x + 8 = 17$, the student must imagine that he can remove the number of nails that Bob found from the total number of nails. This suggests subtracting 8 from 17. The result of this subtraction is the replacement for x that makes a true statement from the condition. The solution set of the condition is $\{9\}$. This is used to obtain the answer to the question asked in the problem.

The problems in D1 and D2 could be classified as additive-subtraction. The real or imagined action in these problems is additive, but the solution of the condition for each problem is found by subtraction. The term "additive-subtraction" should not be used with your students.

The problems in D3 involve finding the original number of objects when the number of objects that are gone and the number of objects that are left are known. A general form of the sentence that expresses the condition for this problem is $n - b = c$. The replacements for the variables b and c are given in the problem. n is a variable for the number of objects there were to begin with.

To find the original number of objects for each of problems E and F in D3, the students imagine that the 9 that were removed are put back with the 15 that remain. They thus see that the replacement for n that makes a true statement from the condition $n - 9 = 15$ can be found by adding 9 to 15. Once again, the solution set of the condition is used to obtain the answer to the question asked in the problem.

Problems E and F in D3 could be called subtractive-addition. The action described in the problem is subtractive, but the solution of the condition for the problem is found by addition.

A fourth problem type introduced in the lesson involves finding the number of objects removed when the original number of objects and the number of objects that remain are known. A general form of the sentence that expresses this type of condition is $a - n = c$. The replacements for the variables a and c are given in the problem. n is a variable for the number of objects that are removed.

In problems G and H in D4, an undisclosed number of objects were removed from 32 objects, and 13 objects were left. The condition $32 - n = 13$ reflects the structure of each problem. To find the number of objects that were removed, the student imagines that he can remove the 13 objects from the original number of objects, 32. He reasons that 32 is made up of n and 13, that is, $32 = 13 + n$. If he takes the other component (13) away from 32, n will be left.

The problems in D4 could be classified as subtractive-subtraction. The action indicated in each problem is subtractive, and the solution of the condition is found by subtraction.

The students have had considerable experience with problems similar to those presented in D5 and D6. The problem in D5 is solved by addition, while the problem in D6 involves subtraction. To answer the questions asked, the student first performs the computation indicated by the condition for each problem.

For several problems in this lesson, the students are asked to verify their answers mentally. This means that they should mentally replace the variable in the condition by the member of the solution set to see if it makes a true statement.

Instruction notes

You might find it helpful to use physical objects to illustrate the action described in each problem of this lesson and to indicate how the solution of the condition for each problem can be found.

Try to ensure that each student in the class understands the problem-solving procedure and the interrelationship of all steps in the procedure. Point out that, because we are primarily concerned with a clear understanding of the problem-solving procedure, the problems are not particularly difficult. The students should use the procedure presented in the lesson, however, even though they know "how to get the answer."

After the class has discussed exercise C on page 69, you might ask your students: "What relation does the solution of the condition have to the question asked in problem A? In problem B?" [The solution is used to obtain the answer to the question.]

When your class has discussed exercise T on page 71, ask: "Why have we subtracted 13 from 32?" [Because 13 is a part of 32, and we want to find the other part.] "Does the condition $32 - n = 13$ indicate the action that is described in problem G?" [Yes] "In problem H?" [Yes] "Does the condition $32 - 13 = n$ indicate the events that are described in problem G?" [No] "In problem H?" [No] "Should you use the condition $32 - 13 = n$ for either problem G or problem H?" [No]

Upon completion of exercise H on page 71, ask the students: "In what way is the condition for the problem in D5 similar to the conditions for the problems in D1 and D2?" [They all involve addition and the idea of equality.] "How is the condition for the problem in D5 different from the conditions for the problems in D1 and D2?" [The solution for the condition used for the problem in D5 is found by using addition; for the problems in D1 and D2, the solution is found by using subtraction.]

After the students have answered the question in exercise N on page 72, have them compare the condition for the problem in D6 with the conditions for the problems in D3 and D4.

Ask questions that lead your students to see that the action described in the problem in D6 is the same as the action described in the problems in D3 and D4. They should note, however, that in D5 and D6 the result of the action gives the group whose number is to be found, which is not true of the problems described in D1, D2, D3, and D4.

Provision for individual differences

■ To provide more practice for the students in your class, have them solve problems 1 through 5 and problems 7 through 10 of the "On your own" section on page 64. They should follow the directions for the "On your own" problems in this lesson.

If your students did not use the *STA* program in earlier grades, they may need more practice in finding solution sets of conditions. You could write on the board sentences that express conditions such as $x - 25 = 43$ and $58 + n = 87$ and have the students tabulate the solution set for each of these conditions, using N as the universe.

1 / 20 Exploring problems / Pages 73-76

Solution sets of conditions for inequality

Objectives

■ The students learn to solve problems by finding the solution sets of conditions for inequality.

Content overview

■ In lesson 18 the students wrote sentences that expressed conditions for inequality for prob-

lems. Now they will learn to find the solutions of such conditions, and they will use the solutions to answer the questions asked in the problems.

"U = C" is introduced in this lesson. The students learn that "U = C" means that the universe is the set of counting numbers. The only difference between set C and set N is that zero is not included in set C. "U = C" is used when we do not want students to consider zero as a possible replacement for a variable.

In lesson 5 the students found solutions of conditions such as $13 + x < 30$ when the universe for the variable was a finite set. Now they learn to relate the solution of a condition for equality such as $13 + x = 30$ to a condition for inequality such as $13 + x < 30$. The solution of $13 + x = 30$ is 17. 17 is one greater than the greatest number in the solution set of $13 + x < 30$. If the condition for inequality were $13 + x > 30$, the least number in the solution set would be one greater than the solution of $13 + x = 30$.

The students consider problems that involve conditions such as $13 + x < 30$ when the universe is an infinite set. They obtain the solution of $13 + x = 30$ in the manner described in lesson 19. Then they test numbers less than and numbers greater than the solution of the condition for equality as replacements for the variable in the condition for inequality. The students learn to test a few replacements and make a generalization about the members of the solution set of the condition for inequality. In the case of $13 + x < 30$, the solution of $13 + x = 30$ is greater than the solutions of $13 + x < 30$. Hence, the solution set of $13 + x < 30$ includes those members of the universe (U = C) that are less than 17.

For most problems in which the condition is a condition for inequality, the solution set of the condition has more than one member. For problems such as the problem in D1, there may be more than one answer to the question.

The conditions for the problems in D2 have infinite solution sets. The problems do not give enough information to provide a finite set of solutions for the condition. The information given in exercise L on page 74 actually supplies a second simple condition for problem C in D2. Now the numbers to be found must satisfy the condition expressed by the compound sentence $x + 12 > 40$ and $x < 36$. In a later lesson, students will learn that a condition such as this one, which is composed of two simple conditions and the connective "and," is a compound condition.

The solutions of a compound condition with the connective "and" must satisfy each of the simple conditions. When both $x + 12 > 40$ and $x < 36$ are considered, and U = N, the solution set of the compound condition is {29, 30, 31, . . ., 35} At this stage, the students think of $x + 12 > 40$ as the condition for the problem and obtain {29, 30, 31, . . .} as the solution set. Then they must use whatever information is given in the problem to select the members of the solution set that can be used to obtain sensible answers.

In D3 another problem is described in which each member of the solution set of the condition cannot be used to obtain an answer to the question asked in the problem. The solution set of the condition is $\{y | y > 17\}$, but only those numbers greater than 17 and less than 25 are possible answers to the question in the problem.

The sentence $24 - y < 7$ reflects the action and the sequence of events in the problem in D3. However, $24 - y < 7$ is not the only condition that the numbers to be found must satisfy. For problems in which the number to be found is either the least number or the greatest number in the solution set of the condition, students must find the solution set of a simple condition for the problem and then choose the member of the solution set that can be used to answer the problem.

nstruction notes

After the introductory paragraphs are read, you ight have the students tabulate finite as well as finite subsets of C. Ask them if {0, 2, 4, . . .} a subset of C. [No] Then have them tabulate e solution set of $x + 6 < 10$ when U = N and ıen U = C. [When U = N, {0, 1, 2, 3}; when = C, {1, 2, 3}]

As you teach this lesson, it may be necessary r you to review some of the work in lessons 18 ıd 19. The use of physical objects may help to ustrate the action that is described or must be ıagined in each problem in this lesson. You can so use objects as markers to help find the solu- ɔn set of each condition.

Following a discussion of developmental exer- se I on page 73, you could pose these questions your class: "You have decided that $\{x \mid x < 17\}$ escribes the same set as $\{x \mid 13 + x < 30\}$. Does e condition $x < 17$ indicate the action for either oblem in D1?" [No] "Does the condition $x < 17$ dicate the sequence of events in either problem D1?" [No] "When you write a sentence that xpresses a condition for either problem in D1, hich do you write, '$x < 17$' or '$13 + x < 30$'?" $3 + x < 30$, since it shows the action and the se- uence of events in either problem.]

Provision for individual differences

If your students are slow learners who did not se *STA 5** and *STA 6*†, you may want to pro- ide more practice in developing conditions for ıequalities. One possible way to do this is to ıoroughly discuss some of the "On your own" roblems in class. For problem 1, you could ask: What is the condition for this problem?" $x + 7 < 100$.] "What condition for equality will help you find the solutions of the condition for inequality?" [$x + 7 = 100$.] "Will each solution of $x + 7 < 100$ be greater than or less than 93?" [Less than] "What is the universe for the variable?" [N] "What members of the universe satisfy the condition?" [0 through 92] "Which member do you use to obtain the answer to the problem?" [92]

You could have the class work in finding solution sets for exercises 19 through 24 and exercises 26 through 36 on page 22. If so, use U = C. More practice could be provided by using "On your own" problems 1, 2, 3, 6, 9, and 12 in lesson 18. Use the directions for the "On your own" problems in lesson 20. U = C.

1 / Applying mathematics / Page 77

■ Throughout *STM 1*, maintenance in problem solving is provided by sets of problems that appear under the heading "Applying mathematics." These sets of problems provide a cumulative review of various kinds of problems and problem-solving procedures.

The problems in the "Applying mathematics" on page 77 includes examples of types of problems presented in lessons 1 through 20. There are nineteen sets of "Applying mathematics" problems in *STM 1*.

Because the teaching procedures suggested here apply, in general, to all of the "Applying mathematics" sections in the book, specific suggestions for each section are *not* provided. However, suggestions are provided for those sets of problems that introduce non-mathematical terms such as *commission*, *discount*, and so on. Separate teaching notes are also provided for the sets of

Maurice L. Hartung, Henry Van Engen, and Lois Knowles, *Seeing Through Arithmetic 5* (Toronto: W. J. Gage Limited, 1963).

Maurice L. Hartung, Henry Van Engen, and Lois Knowles, *Seeing Through Arithmetic 6* (Toronto: W. J. Gage Limited, 1963).

"Applying mathematics" problems that deal with the following: measures associated with weight, capacity, and time; perimeter of a polygon; and measures associated with area.

To use the "Applying mathematics" problems to the best advantage for the abler students, have them solve the problems independently without any prior discussion. When they have finished the entire set, discuss the problems concerning which they have specific questions.

You can help the slow learners by giving them individual help or by a class discussion of each problem after the students have been given time to work on a problem.

Instruct the students to use the problem-solving procedures that are used in the "Exploring problems" lessons. For each problem, require them to write a sentence that expresses a condition for the problem; then to find the solution set of the condition and use the members of the solution set to obtain an answer for the problem.

---

1 / 21 Exploring problems / Pages 78-79

Conditions for abstract problems

Objectives

■ The students learn to solve abstract problems by finding the solution sets for conditions.

Content overview

■ In lessons 19 and 20, students wrote sentences expressing conditions for problems and then obtained answers to the problems by using solutions of the conditions. This lesson develops a transition from word problems about physical situations ("story" problems) to abstract problems that are also described by words.

The transition is accomplished by first presenting a "story" problem and then presenting an abstract problem that involves a similar action with numbers only. The same condition is used for both problems.

Even though the same condition is used for each problem in D1, there is a difference in the way the answers to the questions are stated. The answer for problem A is a number of objects, 17 model cars. The answer for problem B is a number, 17. The answer or answers to the question asked in an abstract problem are the members of the solution set of a condition for the problem. For a "story" problem, the possible answers are obtained from the solutions of the condition.

Note that $16 + x$ is different from $x + 16$. The phrase $16 + x$ means that x is added to 16, but $x + 16$ means that 16 is added to x. We know that, when x is replaced by a natural number, $(x + 16)$ and $(16 + x)$ are the same natural number. This is an instance of the commutative property of addition of natural numbers. This property is developed in unit 6. However, even though addition is commutative for natural numbers, it is important for a student to write a sentence that expresses a condition that reflects *both the action and the sequence of events described in the problem*. Since it is clear that a number is added to 16 in the problems in D1, $16 + x$, not $x + 16$, describes the situation in each problem.

Problems 16 and 17 in the "On your own" problems of this lesson help to prepare students for the study of compound conditions, which are introduced early in unit 2. For each of these problems, students make a simple condition, find its solution set, and select the solution that is the answer to the question in the problem. Note that these two problems differ from the other abstract problems in that only one of the solutions of the condition is an answer to the question.

Instruction notes

■ When you have discussed exercise I, you might ask: "Is the answer to the question in problem B

number, or is it a number of objects?" [A number] "Does the solution set of the condition for problem B contain the answer to the question asked?" [Yes] "Is the answer to the question in problem A a number, or is it a number of objects?" [A number of objects] "Does the solution set of the condition for problem A contain the answer to the question asked?" [No]

To answer an abstract problem, the student can give either the tabulation or a standard description of the solution set. The tabulation should be used whenever possible. The answer to a problem about physical objects can be *obtained* from the members of the solution set.

Review briefly with your class the meaning of each of these terms: *sum*, *difference*, *product*, and *quotient*.

Notice that this is the last lesson in unit 1. Tests 11 through 17 on pages 80 and 81 are end-of-unit tests. Also, in the supplementary section of this *Teaching Guide*, there is a test for unit 1 that you can reproduce for use with your class.

Provision for individual differences

Have your abler students make up other abstract problems. Also, give them open sentences, and have them write problems whose conditions are expressed by these sentences; or select sentences from exercises 33 through 56 of the "Keeping skillful" on page 79 for this purpose.

---

1 / Checking up / Pages 80-81

End-of-unit tests

Each unit in *STM* contains a set of end-of-unit tests. The end-of-unit tests are objective tests of the following types: completion, matching, multiple-choice, objective-response, true-false, and problem-solving. A variety of types is used because some questions can be stated better in one form than in another.

Each item contained in an end-of-unit test is followed by a numeral in parentheses. The numeral gives the number of the page on which the idea tested by the item is developed. This feature aids the students in the use of the tests for review purposes.

If you wish, you can use an end-of-unit test from the student's book as a homework assignment to review the ideas presented in the unit. Tell the students that, if they cannot answer an exercise, they should turn to the page indicated by the numeral in parentheses and restudy the topic. The answers are given in the *Response Verification Books* for *STM 1*.

There are several ways in which you may evaluate the students' learning achievement when you have completed the teaching of a unit. You can use the end-of-unit tests in the student's book; you can construct a test based on the end-of-unit tests in the student's book; or you can use the end-of-unit tests that are given in the supplementary section of this *Teaching Guide*.

You have the publisher's permission to reproduce the end-of-unit tests that appear in the *Teaching Guide*. Answers for the items in these tests are given with the tests. They are not contained in the *Response Verification Books* for *STM 1*. As for all such tests, students should be given sufficient class time to complete all of the test items.

See pages 54 and 55 of this *Teaching Guide* for a description of other ways to administer tests and for suggestions on how to conduct an effective testing program.

Notice that there are no end-of-block tests for the last series of lessons in each unit. Test items that cover the concepts and skills developed in these lessons are included in the end-of-unit tests.

Intersection of sets

Objectives

■ The students learn how to obtain the intersection of two sets and to use Venn diagrams to represent the intersection of two sets.

Content overview

■ Addition and multiplication are operations on numbers. In these operations, two numbers are used to obtain another number. Therefore, addition and multiplication are binary operations. The result of addition is a number with the special name *sum*, and the result of multiplication is a number with the special name *product*.

In a similar way we can think of operations on sets. The intersection of two sets is the result of a particular operation. In this operation we consider those objects, and only those objects, that belong to both sets. The result of this operation is *intersection*. There is no special name for the operation.

The symbol $\cap$ is used to express the idea of intersection. This symbol may be read "cap" or "The intersection of" ("A cap B" or "The intersection of A and B").

From the definition of an operation, it can be determined whether or not the operation has certain properties. For example, when U = N, a true statement is obtained from $a + b = b + a$ for each replacement of the variables. Similarly, when U = N, a true statement is obtained from $ab = ba$ for each replacement of the variables. These are the commutative properties of addition and multiplication, which are studied in unit 6.

Note that in exercise L, page 83, the students are asked whether the set operation that results in an intersection of two sets has the commutativ property. Because $A \cap B = B \cap A$ for any set A and B, this operation does have the commuta tive property.

In the next lesson, the students will study an other operation whose result is called *union*. Th union of two sets is the set that contains all th members of the two given sets.

Venn diagrams, named after John Venn, a English mathematician and logician, are used t help develop the idea of intersection. In these dia grams, relations between sets are illustrated wit the universe usually pictured as a rectangle an subsets of the universe pictured as circular region included in the interior of the rectangle. Thes diagrams are also used in lesson 23 to illustrat the set operation whose result is called union Venn diagrams provide a visual picture of ho sets are related under the set operations.

The following are cases that occur:

1 Two sets may have some members in common $A \cap B = K$, and K is not the empty set.

2 Two sets may have no members in common $A \cap B = \{\ \}$.

3 One set is a subset of another set. If $A \cap B$ = B, then B is a subset of A. If $A \cap B = A$, then A is a subset of B.

The idea of intersection is also used with th work on geometry in lessons 26 through 29.

Instruction notes

■ Explain to your students that in this unit the will learn more about sets, conditions, and ge ometry. In this lesson they will learn about th intersection of sets. Some of the sets they wil talk about will be sets of concrete objects. Other will be sets of numbers. Later in the unit, the stu dents will study the intersection of geometri figures.

Because many new terms and the symbol $\cap$ are introduced in this lesson, you should follow the developmental exercises closely. After exer

se G on pages 82 and 83, point out that the let-rs C and D are not variables. Capital letters sed in this way are names of sets, not names of ariables.

Following exercise S on page 84, say: "When ou found the intersection of sets K, L, and M, id you find the intersection of all three sets at nce, or did you find the intersection of K and L nd then find the intersection of that set with set 1?" [The latter] "Is it possible to find the inter-ection of more than two sets at once?" [No]

After discussing exercise X on page 86, the ollowing procedure may help students to achieve etter understanding of the empty set as the in-ersection of two sets. Write on the chalkboard {1, 2, 3} ∩ {5, 10}." Say: "Suppose that we re thinking about the intersection of these two lisjoint sets. What is their intersection?" [The mpty set] "We have to decide if the empty set is subset of each of these disjoint sets. Turn to the lefinition of *subset* on page 11. Is each member of the empty set a member of the set whose mem-bers are five, ten? This may be difficult for you o think about since the empty set has no mem-bers. But we can say that the empty set is a subset of the set whose members are five, ten because, in saying this, we do not disagree with the definition of subset. In mathematics we often find it useful o think of the empty set as being a subset of every set. Another way of defining a subset is to say that A is a subset of B if A contains no mem-bers that are not members of B."

After exercise Y on this page, say: "Now de-cide if the intersection of the empty set and an-other set is a subset of each of the sets. If you say that each member of the empty set is also a mem-ber of the empty set, do you violate the definition of subset?" [No]

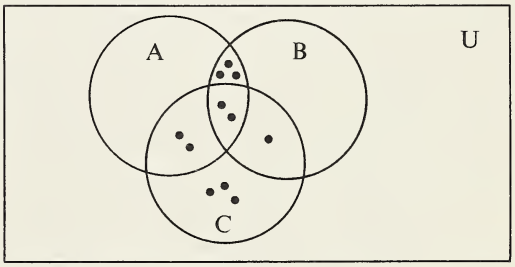
Provision for individual differences

■ Ask your abler students to try to answer the following questions:

1 $A \cap B$ is always a subset of what sets? [A and B] Is $A \cap B$ always equal to $B \cap A$? [Yes]

2 Can an infinite set be the intersection of two finite sets? Explain your answer. [No; a finite set cannot have a subset that is an infinite set.]

3 There are 5 members in the intersection of sets A and B. There are 3 members in the intersection of sets B and C. There are 2 members in $A \cap B \cap C$. There are 3 members in C that are not in $B \cap C$ or in $A \cap C$. There are 4 members in the intersection of A and C. How many members are in C? Make a Venn diagram and use dots to help you find the number of members in C. [8]



Have slow learners find the intersection of sets of familiar objects. If possible, have them work with objects until they achieve understanding of the idea of intersection, disjoint sets, and so on.

You might have the students with blond hair stand; then have the students with blue eyes stand. From these sets, have the students find the members of the set of students who have both blue eyes and blond hair. Have them tabulate sets from given descriptions and encircle the names of objects that are in both sets.

Use Venn diagrams throughout the discussion. Make transparencies that may be used with an overhead projector. Subsets of a universe can be pictured separately and then together, using an overlay to illustrate the idea of intersection.

Union of sets

Objectives

■ The students learn how to obtain the union of two sets and to use Venn diagrams to represent the union of two sets.

Content overview

■ This lesson develops ideas associated with another set operation. The result of this particular operation is called *union*. As for the operation discussed in lesson 22, there is no special name for the operation discussed in this lesson.

The *union* of two sets is the set that contains those members that belong to either set or to both sets. Since only those members contained in both sets form the intersection, the intersection is always a subset of the union of the two sets.

The symbol $\cup$ is used to express the idea of union of sets. This symbol may be read "cup" or "The union of" ("A cup B" or "The union of A and B").

Since all the members contained in either, or in both, of the two sets are in the union, each of the given sets is a subset of the union.

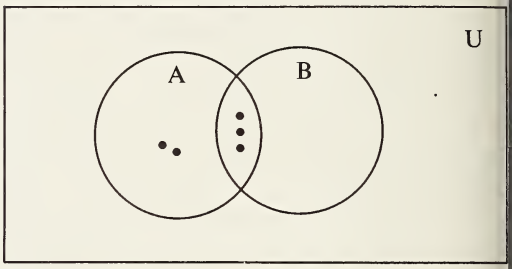
Like the operation whose result is called intersection, the operation whose result is called union is commutative. Note that exercise N on page 87 asks if the operation that results in the union of two sets has the commutative property.

Problems such as the one given below involve the relationship between the number of members in two sets and the number of members in their intersection and in their union.

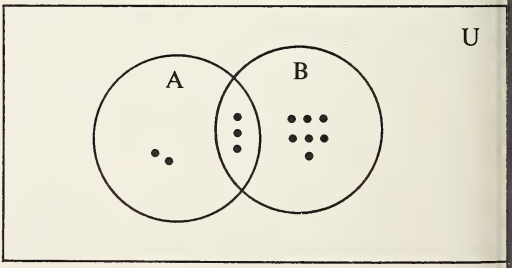
In Don's class, five students study the violin. Twelve students study either the violin or the piano or both. Three students study both the violin and the piano. How many students study the piano?

Suppose that A is the set of students who study the violin and B is the set of students who study the piano. $A \cap B$ is the set of students who study both the violin and the piano. Then $A \cup B$ is the set of students who study either the violin or the piano or both the violin and the piano.

From the information given in the problem, you can make the diagram below. Set A has 5 members and $A \cap B$ has 3 members.



To complete the diagram, you can use the fact that $A \cup B$ has 12 members. Five members are represented in the diagram above. The other 7 members are in B, but not in the intersection of A and B. The completed diagram is shown below.



From the diagram you see that set B has 10 members and that 10 students study the piano.

A general method for solving such a problem can be used. If you subtract the number of common members from the sum of the number of

iembers in the two given sets, you obtain the umber of members in the union of the two sets. you use n(A), n(B), n(A ∩ B), and n(A ∪ B) represent the number of members in A, B, ∩ B, and A ∪ B, respectively, then a true atement is always obtained from the condition xpressed below.

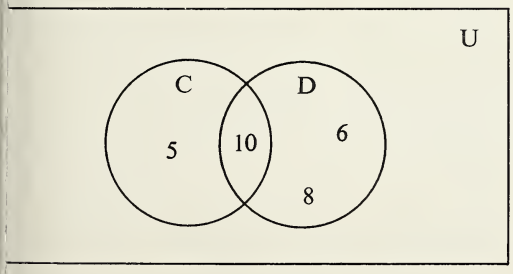
$$n(A) + n(B) - n(A \cap B) = n(A \cup B).$$

For the problem described above, n(A), (A ∩ B), and n(A ∪ B) are known. Therefore, ie condition becomes $5 + n(B) - 3 = 12$. Hence, (B) must be 10.

In lesson 24 of *STM 1* we use the intersection f sets to find solutions of compound conditions hat include the connective "and." In *STM 2* we se the union of sets in finding the solutions of ompound conditions that include the connective or." The union of sets, however, is also used vith the work on geometry in *STM 1*.

Instruction notes

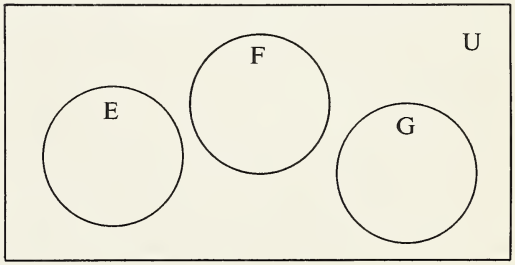
Follow the developmental exercises closely. After exercise L on page 87, say: "Are sets C nd D disjoint?" [No] "Do they meet?" [Yes] "Is one a subset of the other?" [No] "What nember do they have in common?" [10] "The nion of sets can be represented by Venn dia-grams." Sketch this diagram:



"Does this diagram show the relation between sets C and D?" [Yes]

Continue the exercises through U on page 88. Then ask: "When you found the union of sets E, F, and G, did you find the union of all three sets at once, or did you find the union of E and F and then find the union of that set and G?" [The latter] "Let us make a Venn diagram of the sets described in D7. What is the universe?" [N] "Do sets E and F meet?" [No] "Do sets E and G meet?" [No] "Is G a subset of E?" [No] "Do sets F and G meet?" [No]

Sketch a diagram like this one:



"Does the diagram show the relation among sets E, F, and G?" [Yes]

Finish the developmental exercises.

Provision for individual differences

■ Abler students might explore problems similar to the one described in the "Content overview" section of the notes for this lesson. First explain that n(A) means the number of members in set A, n(A ∩ B) means the number of members in the intersection of sets A and B, and so on. Give them specific sets A and B and have them determine n(A), n(B), n(A ∩ B), and n(A ∪ B).

Ask the students to try to develop a method for solving the problems that follow. Tell them to draw Venn diagrams for the problems. Some of your abler students should discover that the condition $n(A) + n(B) - n(A \cap B) = n(A \cup B)$ is a true statement for any two sets A and B.

1 Seventeen families in Joan's block read newspaper A. Twenty families read newspaper B. Thirty families read either newspaper A or news-

paper B or both newspapers. How many families read both newspapers? [7]

2 Twelve students in Jack's class have one or more brothers. Fourteen students have one or more sisters. Six students have both a sister and a brother. How many students have at least one brother, or one sister, or both a brother and a sister? [20]

3 On a test, 83 students answered the first question correctly. Forty students answered both the first and the second questions correctly. Eighty-eight students answered either one or both of the first two questions correctly. How many students answered the second question correctly? [45]

If you work carefully and slowly, utilizing visual aids such as concrete objects and sketches, you will find that slow learners will readily grasp the ideas of intersection and union. You might use mnemonic devices to help them differentiate between the ideas of intersection and union. The symbol for intersection, "$\cap$," is sometimes referred to as "cap." Have the students think of a cap covering only those members common to both sets. The symbol for union, "$\cup$," is sometimes referred to as "cup." Have them think of a cup that contains each member of both sets.

Utilize sets of familiar objects, and give slow learners adequate opportunity to tabulate the union of two given sets.

Compound conditions and their solution sets

Objectives

■ The students learn to find the solution set of a compound condition in one variable involving the connective "and."

Content overview

■ The idea of intersection of sets is used to analyze many algebraic and geometric situations. This lesson relates the idea of intersection of sets with the solution set of a compound condition. In later lessons, the students will write sentences that express compound conditions for problems.

Conditions such as $x + 3 > 7$, $x + 4 = 9$, and $\overline{AX} \cong \overline{CD}$ are simple conditions. Logical connectives such as "and" and "or" may be used with two or more simple conditions to form a compound condition.

A compound condition that includes two simple conditions and the connective "and" is satisfied by only those members of the universe that satisfy each of the simple conditions. The solution set of $x + 3 < 7$ and $x + 4 = 6$ is $\{2\}$ when the universe is N. The number 2 is the only member of N that satisfies both of the simple conditions. To find the solution set of a compound condition, the student can first find the solution of each simple condition. The solution set of the compound condition is the intersection of the solution sets of the simple conditions that make up the compound condition.

The solution set of a compound condition that includes the connective "and" can also be found by considering the solution set of just one of the simple conditions and then determining which of these solutions are also solutions of the other simple condition. Usually it is simpler to consider first the simple condition that has a finite solution set, if there is such a condition. This method is possible because the solutions of the compound condition are also solutions of each of the simple conditions. This implies that the solution set of a compound condition must be a subset of each solution set of the simple conditions that form the compound condition. Venn diagrams help the students understand this relationship among the three solution sets.

Only compound conditions that involve two imple conditions and the connective "and" are onsidered in this lesson. There are other kinds f compound conditions. A compound condition hat includes two simple conditions and the con- ective "or" is satisfied by those members of the niverse that satisfy either one or both of the imple conditions. Hence, the solution set of the ompound condition "$x + 3 = 7$ or $x + 3 = 9$" is 4, 6} when U = N. On the other hand, the solu- ion set of "$x + 3 = 7$ and $x + 3 = 9$" is { }, be- ause no member of N satisfies both simple con- itions. Compound conditions involving the con- ective "or" are considered in *STM 2*.

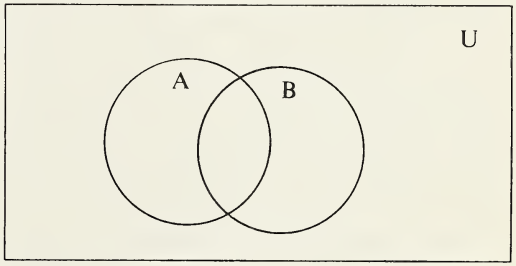
The symbol $\wedge$ expresses an operation of con- unction ("and") between conditions. Since this peration is commutative, $x + 3 > 7 \wedge x + 3 < 9$ has the same solutions as $x + 3 < 9 \wedge x + 3 > 7$. The property of commutativity of the operation f conjunction is implied in exercise F, page 91.

Graphs, as well as Venn diagrams, can be used o help students understand the relationship be- ween the solution set of a compound condition hat involves "and" and the solution sets of the imple conditions in the compound condition. In liscussing graphs, we have used the phrase "dots nclosed by squares." Squares are, of course, nathematical ideas that do not exist in the physi- al world. What the students see on the page is pictures of squares rather than the squares them- elves. As is occasionally the case, we use the briefer form of expression when it is not likely to ead to confusion.

Instruction notes

▪ You might introduce the lesson in the following manner. Say: "To give you an idea of the kind of condition you will learn about in this lesson, we will use the members of the class as the universe. Suppose we have the condition that student x is a member of row one and x wears glasses." Write on the chalkboard "x sits in row one, and x wears glasses." Select a row in which two or more students are wearing glasses and call it row one. "I will now list the names of the persons who sit in row one." Write these names on the chalkboard. "Does each of these persons satisfy the first condition?" [Yes] Write the names of all students in the class who are wearing glasses. "Does each of these persons satisfy the second condition?" [Yes] "What students satisfy both conditions?" Underline the names of these students. "We can make a Venn diagram to show these sets. Let us call the set of students in row one set A and the set of students who wear glasses set B. Are these sets disjoint? Do they meet? Is one a subset of the other?" [Answers depend on class situation.]

Sketch a diagram similar to the one below.



In the portion of the diagram that indicates the intersection of the sets, write the names you have underlined, and write the other names where they belong. We are assuming that set B is not a subset of set A. Ask: "Does this diagram show the relationship between the sets?" [Yes] "Are the people named in the intersection of sets A and B the only people who sit in row one and wear glasses?" [Yes] "Are any students in row one who wear glasses not named in the intersection?" [No] "Is the intersection of sets A and B a subset of set A?" [Yes] "Is it a subset of set B?" [Yes] "Now you will learn how a compound condition is used in situations about numbers."

As you discuss the developmental exercises, be sure students understand that "$\{x \mid x + 3 > 7\} \cap \{x \mid 6 + x < 14\}$" is not the same as "$x + 3 > 7 \wedge 6 + x < 14$." Intersection is the result of an operation on sets, and conditions are not sets.

Provision for individual differences

■ You might ask your abler students to consider compound conditions that involve two "and" connectives and three simple conditions. Some students should arrive at the generalization that each member of the solution set of the compound condition must satisfy each of the three simple conditions. Have the students make up conditions in which the solution set is infinite; in which the solution set is finite but not the empty set; and in which the solution set is the empty set. You could have them make a graph and a Venn diagram for each example.

Slow learners may have difficulty in relating the "overlapping" of the solution sets of simple conditions to obtain the solution set of a compound condition. If so, have them tabulate the solution set of each simple condition, one above the other, and from these tabulations determine the solution set of the compound condition.

Before assigning the "On your own" exercises to your slow learners, you might have them tabulate the solution sets of compound conditions that are involved in social situations such as the one described in the "Instruction notes" for this lesson.

---

2 / 25 Exploring problems / Pages 94-96

Problems involving compound conditions

Objectives

■ The students learn to solve problems that involve compound conditions in one variable.

Content overview

■ This lesson gives students an opportunity to synthesize their knowledge of how to write sentences that express conditions for problems and how to solve compound conditions in one variable.

Many everyday situations involve more than one requirement. A child who is sent to a newsstand for a paper and to a grocery store for a loaf of bread is performing a mission that involves a multiple requirement. Other examples of situations that involve more than one requirement are: completing homework assignments; fulfilling the requirements for an auto operator's license; and buying a home.

In all the problems in this lesson, the action is either additive or subtractive. The compound conditions are of the type that include the connective "and."

The same placeholder is used in expressing each simple condition in a sentence for a compound condition. Since the placeholder in each case refers to the same set of numbers, it is unnecessary and unwise to use two different placeholders. At the present time, your students are not prepared to use a universe for a condition in two variables. In unit 3 they will study compound conditions in two variables.

Instruction notes

■ You can present this lesson by discussing the developmental exercises, or you can have the students read the problems in the displays and then have a general class discussion on writing sentences that express compound conditions for the problems. If you use the latter procedure, students who have difficulty with this lesson should read through the developmental exercises. Plan to spend some class time in discussing the "On your own" exercises, giving particular attention to the sentences the students write to express the conditions in the problems.

Provision for individual differences

Abler students should have little difficulty with his lesson. The "Applying mathematics" problems on page 97 of the student's book may be assigned to these students to provide them with more experience in problem solving.

Problem solving usually is difficult for slow learners. Try to circulate among these students as they work on their own. Check to see that they use the same symbol for a placeholder for each sentence they write to express a simple condition and a compound condition. If a common misunderstanding arises, discuss the troublesome idea with the entire class. If reading difficulties exist, read each problem aloud to the class.

/ 26 Exploring ideas / Pages 98-103

ntersection of lines and planes

Objectives

The students study the intersection of sets of points; that is, they study the intersection of two planes, of a line and a plane, and of two lines.

Content overview

In this lesson the idea of intersection of sets is applied to lines and planes. The intersections developed are related to the ideas of simple geometric objects that are parallel and simple objects that are not parallel.

There is a distinction between the terms *intersection* and *intersects*. For any two sets, the set of common members of the two sets is their *intersection*. If they do not intersect, the intersection is the empty set. *Intersects* refers to specific geometric figures that actually meet; that is, sets whose intersection is not the empty set.

The relations given in the chart in the next column are discussed in this lesson.

Objects two planes	*Relation* parallel intersect	*Intersection* empty set a line
a line and a plane	parallel line included in the plane line intersects the plane	empty set the line a point
two lines	parallel (coplanar) skew (not coplanar) intersect (coplanar)	empty set empty set a point

Instruction notes

■ If you follow the developmental exercises in teaching this lesson, use concrete visual aids such as pieces of tagboard, knitting needles, and the like to represent planes and lines.

The ideas developed in this lesson should be familiar to you if you have studied a basic course in geometry, and you should now be familiar with the symbolism and terminology used in this course. Therefore, you could present the ideas in this lesson in an informal class discussion.

The walls, ceiling, and floor of the classroom are useful models for planes, and the intersections of these planes (walls, ceilings, and floors) can be used to suggest lines in space.

Provision for individual differences

■ For a class of abler students, you could develop the ideas presented in the lesson and have the students read the developmental exercises on their own. The students should be cautioned to give close attention to the properties that are stated and to become familiar with the terms and definitions that are introduced. The "Keeping skillful" exercises on page 103 can be used for review.

With a class of slow learners, plan to spend considerable time using concrete visual aids as

you present the ideas in the lesson. Let your students handle the objects. Then discuss the developmental exercises with your class.

You may want to devote two days to this lesson. If so, you might stop with the first exercise F on page 101 and have the students answer the "On your own" exercises 1 through 5, 8, and 9. Finish the lesson on the second day.

Students might be interested in making models that suggest intersections.

---

2 / 27 Exploring ideas / Pages 104-108

Rays and angles

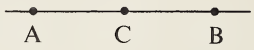
Objectives

■ The students learn what is meant by rays and angles. They also study right angles, linear pairs of angles, and perpendicular lines.

Content overview

■ The idea of ray is developed first. From the definition of ray, it follows that a ray is the union of two sets. A *ray* is the union of the set of points in a half-line and the set that is composed of one point, the boundary of the half-line.

Consider line AB and a point C in line AB. See the diagram in column 2. C separates line AB into two half-lines. Each of these half-lines is a set of points, neither of which contains C. Now consider the half-line that contains B. We say that this half-line does *not* have a left endpoint. This sounds intuitively false, but if we accept a left endpoint for this half-line, it would be a point *immediately* to the right of point C, and there would be no point between C and this endpoint. But there are infinitely many points between any two points in a line. Therefore, should your students question whether or not a half-line has an endpoint, you can answer "No" and justify your answer by this type of reasoning. Ray CB is the union of the half-line that contains B and {C}.



A point in a line that is the boundary of two half-lines determines two rays that are subsets of the line. Of course, any given point in a plane or in space is the endpoint of an infinite set of rays.

The idea of an angle also involves the idea of the union of two sets. The two rays that form the union must have a common endpoint, and they must not be subsets of the same line.

The traditional "straight angle" is not considered an angle in the geometry being developed. If an angle could have a measure of 0° or 180°, it would have no interior or exterior. Also, certain theorems that follow later would not apply to all angles if we accepted "180-degree angles." In lesson 29 the interior and the exterior of an angle are defined.

$\overrightarrow{AB} \cup \overrightarrow{AC}$ is angle BAC, providing that A, B, and C are noncollinear. Since $\overrightarrow{AB}$ is an infinite set of points, it is not a member of $\angle BAC$; it is a subset of $\angle BAC$ because each member of $\overrightarrow{AB}$ is also a member of $\angle BAC$.

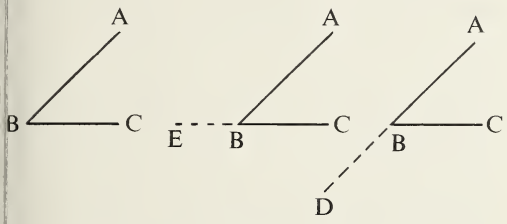
The way in which rays and angles are named is important. The endpoint of a ray is named first, and then any other point in the ray is named. If A and B are different points, then $\overrightarrow{AB}$ is different from $\overrightarrow{BA}$. The symbol for a ray always points to the right. For an angle, the name of the vertex is the middle letter of the three letters used to name the angle. If A, B, and C are noncollinear points, $\angle ABC$ is different from $\angle BAC$.

The set of all linear pairs of angles is a subset of the set of all supplementary angles. Supplementary angles are a pair of angles the sum of whose measures is 180 degrees, but notice that the idea of supplementary angles is developed with-

ut the use of the idea of angle measure. The ietric geometry that includes angle measure is eveloped in unit 8 of the *STM* program.

The set of linear pairs of angles is also a subset f the set of adjacent angles. Adjacent angles are pair of angles that have a common vertex and a ommon side "between" them. A mathematically ound definition is given in unit 8. The term "adacent angles" is not used in this lesson.

The idea of right angle is developed without eference to angle measure. Any angle is a mem- er of two different linear pairs of angles. For xample, $\angle ABC$ pictured below is the supplement f $\angle EBA$ and is also the supplement of $\angle DBC$.



Since supplements of the same angle are congruent, the two supplements, $\angle EBA$ and $\angle DBC$, re congruent. A right angle is an angle that is ongruent to its supplement. When the idea of upplement is generalized, this definition of right ngle still holds. A linear pair of angles is pictured below. $\angle EFG$ is congruent to its supplement, $\angle HFG$. Therefore, $\angle EFG$ is a right angle. By the same reasoning, $\angle HFG$ is a right angle.

The definition of perpendicular lines requires hat all four angles formed by two intersecting lines be right angles. In this lesson, the idea of "perpendicular to" has been applied to lines only. In unit 8, the relation of "perpendicular to" is applied to line segments (see page 474). Unlike the relation of "parallel to," the relation of "perpendicular to" is not transitive. That is, if ℓ_1 is perpendicular to ℓ_2, and ℓ_2 is perpendicular to ℓ_3, then ℓ_1 is *not* perpendicular to ℓ_3. In fact, under these circumstances, ℓ_1 is either parallel to ℓ_3 or ℓ_1 and ℓ_3 are skew lines. The relation of "perpendicular to" is not reflexive, because a line cannot be perpendicular to itself. However, this relation is symmetric. If ℓ_1 is perpendicular to ℓ_2, then it follows that ℓ_2 is also perpendicular to ℓ_1.

The next two lessons include work based on an understanding of rays and angles.

Instruction notes

■ For demonstration purposes, use chalkboard pointers, pencils, or dowels to represent rays. Objects from a wooden building toy set may also be used. Put a thumbtack in one end of the stick or use a similar arrangement with other objects to represent the endpoint of the ray. When you represent lines, be certain the students understand that there is no endpoint. When you make sketches on your chalkboard for this lesson, use colored chalk to suggest the endpoint of a ray.

Provision for individual differences

■ A class of abler students should be able to complete this lesson in one period. You may wish to have them make posters containing pictures of some of the geometric figures studied in the lesson, with each figure named and defined. Other exercises that abler students could consider are listed below:

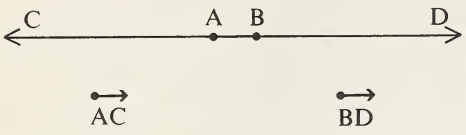
1 Sketch each geometric figure and name the rays:

(a) Two points determine how many rays? [2]

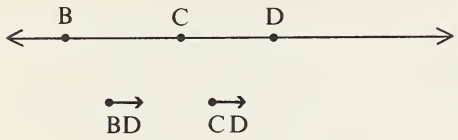
(b) Three noncollinear points determine how many rays? [6]

2 Sketch each of the following:

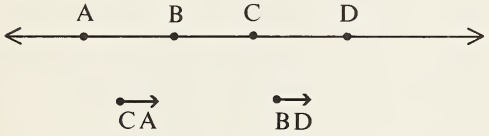
(a) Two rays that are subsets of the same line and whose intersection is the empty set



(b) Two rays that are subsets of the same line and whose intersection is one of the rays



(c) Two rays that are subsets of the same line and whose intersection is a line segment



3 Lines 1, 2, 3, and 4 are coplanar lines. Make a sketch for each of the following:

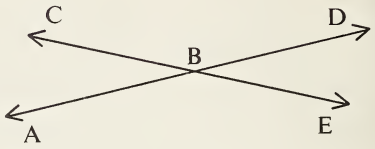
(a) Line 1 and line 2 are each perpendicular to line 3. What is true about line 1 and line 2? [Line 1 and line 2 are parallel.]

(b) Line 1 is perpendicular to line 2; line 3 is perpendicular to line 4; and line 2 and line 4 are parallel lines. What is true about line 1 and line 3? [Line 1 and line 3 are parallel.]

(c) Line 1 is perpendicular to line 2. Line 1 is parallel to line 3. What is true about line 2 and line 3? [Line 2 is perpendicular to line 3.]

4 What is the intersection of two angles that form a linear pair of angles? [The ray that is their common side]

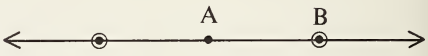
5 Can two pairs of linear angles have an angle in common? [Yes] Make a sketch to explain your answer.



[$\angle CBD$ is common to the linear pair $\angle ABC$ and $\angle CBD$ and the linear pair $\angle CBD$ and $\angle DBE$.]

6 Two intersecting lines form how many linear pairs of angles? [4] Make a sketch to help you explain your answer, and name each pair.

7 When $U = \overleftrightarrow{AB}$, what does $\{X \mid \overline{AX} \cong \overline{AB}\}$ describe? [A set that contains two points]



Slow learners should be given considerable practice in drawing and naming rays, angles, linear pairs of angles, right angles, and perpendicular lines. They should also be encouraged to make sketches to help them find the answers to the "On your own" exercises.

With slow learners you may want to devote two days to this lesson. If so, stop with developmental exercise J on page 106 and "On your own" exercise 11. Finish the lesson on the second day.

The triangle

Objectives

■ The students study another important geometric figure, the triangle.

Content overview

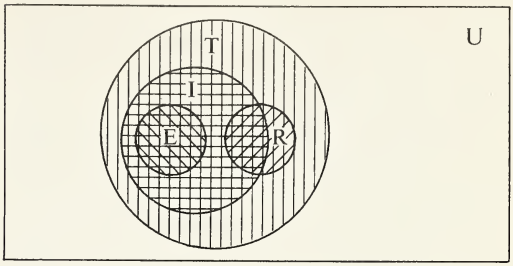
A *triangle* is the union of the three segments at are determined by any three noncollinear oints. The segments are the sides of the triangle. he three noncollinear points are the vertices of ie triangle. The sides of the triangle are subsets f the triangle, and the three noncollinear points e members of the triangle. Since three noncolinear points determine exactly one triangle, the ames of these points, or vertices, can often be sed to name the triangle.

A triangle is a plane figure. If A, B, and C are oncollinear points, they are contained in exactly ne plane. Each pair of these points determines a ne that is included in this plane because, if a line ontains two points in a plane, the line is included in the plane. Since the three lines determined by A, B, and C are included in the same lane, the three segments determined by these oints are also included in that plane. Hence, the riangle determined by A, B, and C is included in ist one plane. This plane may be named plane ABC if one chooses.

Triangles can be classified by the number of ongruent sides they have. When no two sides of triangle are congruent, the triangle is *scalene*. When two sides are congruent, the triangle is *isosceles*. When all three sides of a triangle are ongruent, the triangle is *equilateral*. If a triangle has three sides that are congruent, then it also has wo congruent sides. Therefore, an equilateral riangle is a special kind of isosceles triangle, and he set of equilateral triangles is a subset of the et of isosceles triangles.

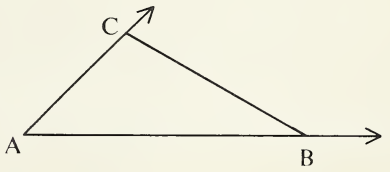
A triangle that has one right angle is a *right riangle*. A triangle cannot have more than one ight angle. Every right triangle is also either scaene or isosceles, but cannot be equilateral.

The Venn diagram in the next column represents the relation among these different kinds of triangles.



In the drawing, U is the set of all geometric figures; T is the set of all triangles; I is the set of all isosceles triangles; E is the set of all equilateral triangles; and R is the set of all right triangles. All triangles that are not isosceles triangles are scalene triangles. Notice that equilateral triangles are also isosceles, but not all isosceles triangles are equilateral.

Notice that, in the middle of the second column on page 109, we explain what is meant by "angles of a triangle." In $\triangle ABC$, pictured below, $\angle CAB$ is not a subset of the triangle, because the angle contains points that are not in the triangle. Despite this, $\angle CAB$ is called "an angle of $\triangle ABC$." Of course, $\triangle ABC$ is not a subset of $\angle CAB$, because the angle does not contain the points between B and C that are in the triangle.



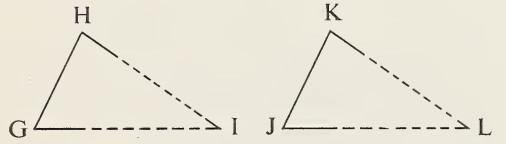
A comparison of the six parts (three sides and three angles) of one triangle with the six parts of another triangle leads to the definition of congruent triangles. In the geometry studied by the students so far, the idea of congruence has meant "the same size," and the term "congruent" was applied to both segments and angles. For trian-

gles, congruence means "the same size and shape." When each part of one triangle can be paired with a part of another triangle so that each pair of angles and each pair of sides are congruent, the two triangles are congruent.

Actually, the definition above makes more than the necessary requirements for congruent triangles. It is possible to determine that two triangles are congruent if you know that three parts of one triangle are congruent to three parts of the other triangle. However, you can do this only with certain groups of three parts. See the diagram below. Two sides and the "included" angle of $\triangle ABC$ are congruent respectively to two sides and the included angle of $\triangle EDF$. (An angle is the "included" angle of two sides of a triangle if the two sides are subsets of the angle.) From this information, you know that $\triangle ABC \cong \triangle EDF$.

$\overline{AB} \cong \overline{ED}$. $\angle ABC \cong \angle EDF$. $\overline{BC} \cong \overline{DF}$.

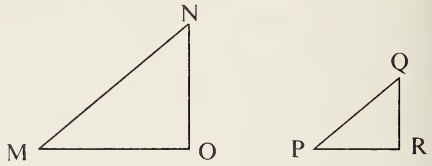
See the diagram below. Two angles and the "included" side of $\triangle GHI$ are congruent respectively to two angles and the "included" side of $\triangle JKL$. (A side is the "included" side of two angles of a triangle if the side is a subset of each angle.) From this information, you know that $\triangle GHI \cong \triangle JKL$.



$\angle G \cong \angle J$. $\overline{GH} \cong \overline{JK}$. $\angle H \cong \angle K$.

Another requirement for congruence of triangles involves just the sides. If three sides of one triangle are congruent respectively to the three sides of another triangle, then the two triangles are congruent.

There is no condition for congruence of triangles that involves only the angles. See the diagram below. The angles of $\triangle MNO$ are congruent respectively to the angles of $\triangle PQR$. However, the triangles are not congruent.



When you know that two triangles are congruent, you know that the six parts of one triangle are congruent *respectively* to the six parts of the other triangle. You must, however, be sure that you pair the parts in the correct order. The students will learn more about congruent triangles in unit 8 of the *STM* program.

Instruction notes

■ You might use the following procedure in presenting this lesson. Make three dots for noncollinear points on the chalkboard. Say: "What kind of points do these dots suggest?" [Noncollinear] "How many segments are determined by three noncollinear points?" [Three]

Label the dots A, B, and C. Sketch pictures of the segments that are determined. Ask: "What are the names of these three segments?" [$\overline{AB}$, $\overline{BC}$, and $\overline{CA}$] "Is each segment a set of points?" [Yes] "What is the union of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$?" [Triangle ABC] "Is each of the three points that determines the segments a member of the triangle?" [Yes] "Each of these endpoints is a vertex of the triangle. Each of the segments is a side of the triangle. How many sides does a triangle have?" [Three] "How many vertices does a triangle have?" [Three]

"Think of the three points A, B, and C. Remember that they are noncollinear. How many planes contain three noncollinear points?" [One] Are any of the points between points A, B, and C in a plane different from plane ABC?" [No] How many planes include a given triangle?" [One]

Continue in this way, and finish the lesson.

Provision for individual differences

For a class of abler students, you could develop the ideas in this lesson by using the displays, sketches on the chalkboard, or transparencies with an overhead projector. You could then have your students study the developmental exercises independently. In any case, be sure to have students carefully consider the exercises that are labeled with red question marks.

For a class of slow learners, have the students make sketches of objects discussed as you develop the ideas in the lesson. On the first day, you might stop with exercise R on page 111 and complete "On your own" exercises 1 through 21.

If your class is not able to maintain the work schedule you have arranged, you could omit the part of the lesson dealing with congruent triangles. If you decide to do this, do not discuss displays 5 and 6 and developmental exercises A through G on page 111. Also omit "On your own" exercises 22 through 26. Congruence for triangles is developed in detail in lesson 105.

2 / 29 Exploring ideas / Pages 113-117

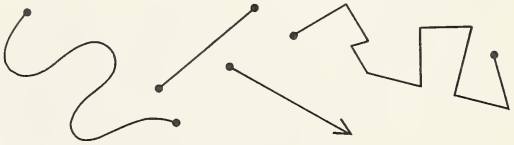
Regions of the plane

Objectives

■ The students learn that angles and triangles separate a plane into two regions. They also study simple closed curves and simple polygons.

Content overview

■ Although the ideas presented in this lesson are grouped under the unifying topic "Regions of the plane," the term "region of a plane" is not defined in this lesson. A definition of this term would involve the idea of simple open curves, with which the students are not familiar. It is a difficult concept to introduce, and we do not wish to consider simple open curves with students at this time. A simple open curve is an infinite set of points that is a subset of a plane. Some simple open curves are pictured below.



Notice that a subset of a line is considered to be a simple open curve and that a simple open curve does not intersect itself.

The students have learned in previous lessons that a line separates a plane into two half-planes and that a circle separates a plane into the interior of the circle and the exterior of the circle. We utilize these ideas to introduce interior of an angle, exterior of an angle, interior of a triangle, and exterior of a triangle.

The idea of a simple closed curve is introduced in this lesson. A simple closed curve is usually taken as an undefined concept. In the physical world, a representation of a simple closed curve can be sketched if the following requirements are observed:

1 Any "point" of the sketch may be used as a starting "point."

2 The sketch must be completed without lifting the pencil tip from the paper.

3 The sketch must start and stop at the same place.

4 Except for the starting "point," the pencil tip touches each "point" in the sketch exactly once.

5 The completed sketch has exactly one "interior."

A simple closed curve separates a plane into two regions. One of the regions is the interior of the simple closed curve, and the other region is the exterior of the simple closed curve. Each region is a set of points. Any two points in the plane that are not contained in the simple closed curve are either in the same region or in different regions. Pictures A, B, C, D, E, and F in D4 on page 115 of the student's book represent simple closed curves.

The union of a simple closed curve and its interior is a closed region; that is, it is "bounded on all sides." The union of a simple closed curve and the exterior of the curve is an open region. If a simple open curve separates a plane into two regions, the union of the curve and either region is also an open region. We think of lines and angles as simple open curves. Thus, the union of an angle and either its interior or its exterior is an open region.

A simple polygon may be defined as the union of segments determined by coplanar points such that (1) no point is in more than two sides of the simple polygon; (2) each endpoint of a side of the simple polygon is an endpoint of exactly two sides; and (3) any point in two sides of the simple polygon is an endpoint of each of the two sides.

The segments represented in figure 1 at the top of the next column do not form a simple polygon because none of the requirements expressed in the above definition is satisfied. The segments pictured in figure 2 do not form a simple polygon because requirements 1 and 2 are not satisfied. The segments represented in figure 3 do not form a simple polygon because requirement 3 is not satisfied. The union of the segments represented in figure 4 is a simple polygon.

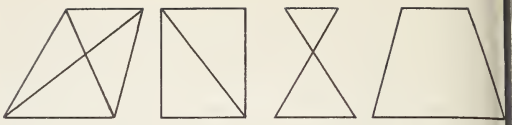


figure 1 figure 2 figure 3 figure 4

Turning now to the subject of logic, the contrapositive of an "if-then" statement is obtained by making both the "if" clause and the "then" clause negative and interchanging the clauses. An "if-then" statement and its contrapositive are either both true or both false.

In exercise E on page 115, the students are actually asked to state the contrapositive of the property stated on page 115, but we do not use this term with the students. This property may also be stated "If a plane geometric figure is a simple closed curve, then it separates the plane into two regions." The contrapositive of this property is: "If a plane geometric figure does not separate the plane into two regions, then it is not a simple closed curve."

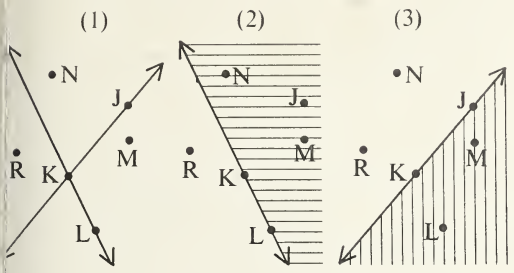
The statement "If a plane geometric figure does not separate the plane into two regions, then it is not a simple closed curve" does not mean that all figures that separate the plane into two regions are necessarily simple closed curves. Some simple open curves, such as a line and an angle, separate a plane into two regions, each of which is an open region.

Angles, triangles, circles, and simple polygons, which are introduced in this lesson, are geometric figures that separate a plane into two regions. The ideas of the area of a parallelogram and of a triangle, which are developed in unit 8, are based on the ideas of the union of a simple closed curve and its interior.

Instruction notes

■ Follow the developmental exercises closely. If you do not discuss each exercise with your class,

)e sure that you present each idea that is devel-)ped. You will find an overhead projector to be xtremely useful in presenting the idea of the in-erior of an angle. Make three transparencies, as hown below:



Project transparency 1, which shows the lines pictured in D2 on page 114 of the student's book. Then project transparency 2, which shows LK and calls attention to the half-plane that contains point M. Now project transparency 3, carefully placed on transparency 2. Notice that "point" K in transparency 1 and "point" K in transparency 2 represent the same point. From this overlay of transparencies 2 and 3, the students should "see" that the interior of angle JKL is the intersection of two half-planes.

Provision for individual differences

■ Abler students should be able to complete this lesson in one day.

Plan to devote two class sessions in teaching this lesson to a group of slow learners. On the first day, stop with developmental exercise T on page 115 and assign "On your own" exercises 1 through 10. Then finish the lesson on the second day.

Have your slow learners practice sketching simple closed curves, but be sure they follow the requirements for making such sketches. This activity should help them to better understand this concept. Discuss with these students the "Keeping skillful" exercises on page 117.

2 / Special challenge / Page 118

Objectives

■ The students apply what they know about simple closed curves by trying to solve a problem.

Content overview

■ The notion of a simple closed curve enters into a solution of a problem in this "Special challenge." A simple closed curve is taken as an undefined concept. In the physical world, a representation of a simple closed curve may be sketched with a pencil if, without lifting the pencil tip from the paper, the sketch of the curve starts and stops at the same place and the pencil tip touches each "point" in the sketch exactly once.

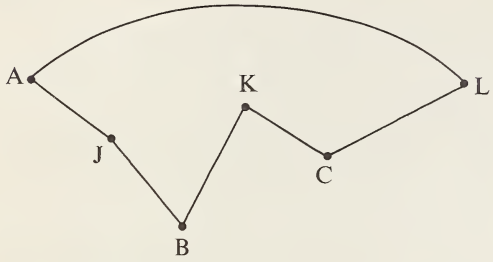
We introduce the idea of a simple closed curve in lesson 29. We develop the idea that a simple closed curve separates the plane into two regions, the interior of the simple closed curve and the exterior of the simple closed curve. The curve is the boundary of each region and is not included in either the exterior or the interior.

Basic to the solution of the problem posed by this "Special challenge" is the idea that a segment determined by one point in the interior of a simple closed curve and one point in the exterior must intersect the simple closed curve. A solution is in the "Responses to Special Challenges" section of this *Teaching Guide*. Another solution is given in the paragraphs that follow.

Think of the pipelines (or connections) from the farmhouses to the wells as segments or as parts of a curve. A point for a farmhouse and a point for a well determine one connection. We can refer to a connection as AJ, CK, and so on.

The following sketch represents a simple closed curve, AJBKCL, that includes six of the required

nine connections: AJ, JB, BK, KC, CL, and AL. The remaining three connections are AK, BL, and CJ.



We can connect A and K in the interior of AJBKCL. We will represent a connection in the *interior* of the simple closed curve by sketching a picture of part of a curve *above* the letters used to identify the curve. For example, the connection AK in the interior of the simple closed curve AJBKCL is represented as $\overset{\frown}{AJBK}CL$.

To connect B to L we cannot, now, make the connection in the interior of AJBKCL. We must connect B and L in the exterior of the curve, and we will represent a connection in the *exterior* by sketching a picture of part of a curve *below* the letters used to identify the curve. Therefore, the connection BL is represented as $AJB\underset{\smile}{KCL}$.

Connection AK in the interior of the curve and connection BL in the exterior of the curve are together represented as $\overset{\frown}{AJB}\underset{\smile}{KCL}$. As you can now see, it is impossible to connect C and J without intersecting the part of the curve that represents the connection AK or the part of the curve that represents the connection BL. Therefore, it is not possible for the farmers to arrange the pipes under the conditions given in the problem.

Instruction notes

■ The eleven groups of "Special challenge" exercises in *STM 1* are intended primarily for the abler students and may be used to provide enrichment for these students. Because the following suggestions apply to each of these groups of exercises, these suggestions are not included in the notes for each "Special challenge." You will find the specific comments for each set of "Special challenge" exercises very helpful.

Have abler students work on each "Special challenge" independently and without prior discussion. Encourage these students to try to answer the exercises without any assistance and to develop a method of explaining the solution they obtain.

Remember that the answers for the "Special challenge" excercises are given in the supplementary section of this *Teaching Guide*, but they are not included in the *Response Verification Books* for *STM 1*. When abler students complete a "Special challenge," either have them check their own answers or discuss their answers with them while the other students in the class are working independently.

---

2 / Checking up / Pages 120-121

Cumulative tests

■ Unit 2 and each unit thereafter terminates in a set of cumulative tests. They are objective tests of the following types: completion, matching, multiple-choice, objective-response, true-false, and problem-solving. The use of a variety of types adds to the reliability of the tests.

Each item in a cumulative test, as is true of each item in an end-of-unit test, is followed by a numeral in parentheses. The numeral gives the page on which the idea tested by the item is developed.

All or part of a cumulative test may be used as a review homework assignment. All or part of a cumulative test may be used to help evaluate the students' retention of learning. Remember that

ıe answers to these test items are given in the *Response Verification Books* for *STM 1*.

If you wish, you can prepare your own cumuative test. Probably the easiest way to do this is o write items similar to the items that are given n the tests in the book. For example, an item imilar to exercise 8 on page 120 could be expressed "$\overline{DE}$ is a segment whose endpoints are ——." An item similar to exercise 14 could be xpressed "$x < 9 \wedge x < 2$."

For specific suggestions on how to conduct an ffective testing program, see pages 54 and 55 of his *Teaching Guide*.

However you use the cumulative tests, we suggest that you do not ignore them. You will find it profitable to have the students refresh their memories occasionally by taking a test on fundamental ideas from preceding units.

3 / 30 Exploring ideas / Pages 122-124

Ordered pairs

Objectives

■ The students learn what is meant by ordered pairs and use ordered pairs to make replacements for two variables.

Content overview

■ In unit 1 we developed methods of finding solution sets of conditions in one variable. For each of these conditions, we used a set of objects as the universe. Conditions that have two variables are considered in unit 3.

The sentence "*m* is the capital of *n*" expresses a condition (non-mathematical) in two variables. For such a condition, a pair of objects is used to replace the variables. These pairs are obtained by using two sets of objects. Each member of one set is paired with each of the members of the other set. The pairs of objects are ordered pairs because the replacements for the variables are made in a particular order.

In the first example in the lesson, each replacement for *m* is a first component of an ordered pair, and each replacement for *n* is a second component. This agreement happens to follow the actual order in which the variables occur in the condition "*m* is the capital of *n*." However, for other conditions, the replacements for the first variable (as it appears in the condition) may be obtained from the second component of each ordered pair; the replacements for the second variable may be obtained from the first component of each ordered pair. For a given condition and universe, the order in which the replacements are to be made are indicated.

In lesson 31 we show how to generate a set of ordered pairs from two sets of objects. Then we use the set of ordered pairs as the universe for a pair of variables in a condition.

Notice that in lesson 30 we say that an ordered pair is used to make replacements. This wording was chosen because the ordered pair itself is not the replacement. The components of the ordered pair are the actual replacements for the variables.

Ordered pairs of objects are fundamental to a great deal of modern mathematics. In this unit we use a set of ordered pairs as the universe for a condition in two variables. In unit 4 the students will study rate pairs, which are a special kind of ordered pair.

In unit 6 the set of natural numbers is developed as a mathematical system. The operations of addition and multiplication are based on ordered pairs, as is any binary operation. In unit 7, where the set of rational numbers of arithmetic is shown to be a mathematical system, the idea of an ordered pair is used again.

Instruction notes

■ You can introduce the lesson in the following way. Write on the chalkboard "$x + 5 = 12$. U = N." Say: "So far you have learned to find solution sets of conditions in one variable. In this unit you will learn to find solution sets of conditions in two variables." Write the sentence $x + y = 12$ on the chalkboard. Ask: "What number satisfies $x + 5 = 12$?" [7] "If I replace x by 7 in $x + y = 12$, do I obtain a true statement?" [No. You obtain another condition.] "In this lesson you will learn about conditions in two variables and about a special kind of universe that is needed for a condition in two variables." Then proceed with the developmental exercises.

Some students may experience difficulty with exercise 9 of the "On your own" exercises on page 124. In this exercise the sentence $y > x + 1$ presents the student with the first example involving a sentence containing the placeholders x and y in which the y precedes the x. Careless students will use members of set C as replacements for y. Unless the thinking of students is clarified on this point, difficulties may be encountered later in this unit. Emphasize to students that the "order" of the placeholders in a sentence does not determine which component of an ordered pair is to be used as a replacement. They should understand, for example, that $x > y$ means the same as $y < x$.

Provision for individual differences

■ At this point in the program, abler students should have an excellent background for the study of conditions in two variables. They are familiar with the notion of conditions in one variable and have had some experience in finding solution sets of conditions in one variable. They have used this experience to solve problems involving conditions in one variable. Also, they are now familiar with much of the symbolism that is used in *STM 1*.

Because abler students retain ideas and skills much better than slow learners do, you will find that the pace you have set for a class of abler students can often be accelerated. You will find, too, that abler students can study much of the material in unit 3 on a self-study basis.

With some classes you may find it possible to cover more than one lesson per day. For example, lessons 30 and 31 could be covered in one day's work with a class of abler students.

Do not attempt to move through the unit at too rapid a pace when working with slow learners. Utilize every possible opportunity to review material presented in the first two units. Try to set a pace that the students find comfortable and one that permits you to give them individual help. Slow learners are much more receptive to new ideas if they are not frustrated.

---

3 / 31 Exploring ideas / Pages 126-130

Sets of ordered pairs

Objectives

■ The students learn to generate sets of ordered pairs and to use the members of these sets to make replacements in conditions in two variables. They find solution sets of the conditions.

Content overview

■ The idea of a set of ordered pairs, which was introduced in lesson 30, is extended in this lesson.

When each member of a set A is paired with each member of a set B, each ordered pair obtained is a member of a third set. This third set is the Cartesian set produced from sets A and B.

Cartesian sets are named after a French mathematician, René Descartes, who worked with ordered pairs of numbers and related these pairs to points in a plane.

A Cartesian set can be generated from one set r from two different sets; it may be a finite or an ıfinite set; and, if two sets are used to form the 'artesian set, they may or may not have the same umber of members. A × B (read "A cross B") is ıe same set as B × A only when A and B are the ame set or either A or B is the empty set. If ither A or B (or both A and B) is the empty set, ıen A × B = { } and B × A = { }. If set A and et B are different sets, A × B and B × A are dif- erent sets, but A × B always has the same num- ber of members as B × A.

The first components of the members of A × B re obtained from set A, and the second compo- ıents of the members are obtained from set B. The first components of B × A, however, are ob- ained from set B, and the second components re obtained from set A. Thus, given a Cartesian et, for example, A × B = {(1, 3), (1, 4), (2, 3), 2, 4)}, it is possible to tabulate set A and set B. For the example above, set A = {1, 2}, and set 3 = {3, 4}.

Not all sets of ordered pairs are Cartesian sets. The set of first components of the members of (1, 2), (2, 1), (1, 1)} is {1, 2}. The set of second components is also {1, 2}. If a Cartesian set is generated from {1, 2} by using each member of {1, 2} as the first component of a set of ordered pairs and each member of {1, 2} as the second component of the set of ordered pairs, the set {(1, 1), (1, 2), (2, 1), (2, 2)} is obtained. Hence {(1, 2), (2, 1), (1, 1)} is not a Cartesian set.

Notice that a sentence such as "The universe for (x, y) is A × B" indicates that the replace- ments for x (in a condition) are members of set A, and the replacements for y are members of set B. If it is intended that x is to be replaced by the second components of A × B, then the sentence must read "The universe for (y, x) is A × B."

Because of the convention that, in a sentence, signs of operation (+, −, etc.) take precedence over signs of relation (=, <, etc.), $m < 2 + n$ in D16 on page 129 means $m < (2 + n)$ rather than $(m < 2) + n$.

You may wish to consult the *Response Verification Book* for answers to "On your own" exercises 18 and 19 before discussing them with your class.

Instruction notes

■ The following questions could be used in presenting this lesson. After exercise A on page 126, say: "Suppose we replace a in this condition by Labor Day. Do we obtain 'Labor Day is a holiday in b'?" [Yes] "Is 'Labor Day is a holiday in b' either a true or a false statement?" [No. It is another condition.] "Both variables in a condition must be replaced to obtain a statement."

Before exercise E on page 126, write "(Thanksgiving)" on the chalkboard. Say: "Is this a symbol for a member of A cross B?" [No. A × B is a set of ordered pairs and (Thanksgiving) is not an ordered pair.] Then write on the board "(Labor Day, June)." Ask: "Which is this, the name of one member or of two members of A × B?" [One] "Remember that, even though an ordered pair has two components, the ordered pair itself is just one member of a set."

After exercise D on page 128, write on the board "(5, 4)." Say: "Is the ordered pair expressed on the board a member of X cross Y?" [No] "Is it a member of Y cross X?" [Yes] "Is X cross Y the same set as Y cross X?" [No] "Is 4 times 5 equal to 5 times 4?" [Yes] "Does X cross Y have the same number of members as Y cross X?" [Yes] Finish the developmental exercises and assign the "On your own" exercises.

Provision for individual differences

■ Have the abler students consider the following:

1 If set A has 5 members and set B has 7 members, how many members are in A × A? B × B? A × B? B × A? [25, 49, 35, 35]

2 Suppose that $A \times B$ and $B \times A$ have common members. What is true about the intersection of A and B? [These common members belong to the intersection.]

3 There are two members in the intersection of sets A and B. How many members are in the intersection of $A \times B$ and $B \times A$? [4]

4 Which of the following are names of Cartesian sets? (a) {1, 2}; (b) {(1, 1)}; (c) {(3, 4), (3, 3), (4, 3)}; (d) {(3, 0), (0, 0), (3, 3), (0, 3)}. [b and d]

5 (2, 3) and (6, 4) are members of a Cartesian set. Name two other ordered pairs that must be in this set. [(2, 4), (6, 3)]

6 What is the Cartesian set of {1, 2} and the empty set? [There are no members in the empty set to match with the members of {1, 2}. Therefore, ordered pairs cannot be formed, and the Cartesian set of {1, 2} and the empty set is the empty set.]

Before assigning the "On your own" exercises, give the slow learners more practice in forming Cartesian sets from sets such as $A = \{5, 7\}$, $B = \{9, 11\}$, and $C = \{0, 4, 6\}$. Have them name each Cartesian set by using a symbol such as $A \times B$.

As usual, give the slow learners individual attention as they work the "On your own" exercises. Be sure to discuss exercises 18 and 19 with these students, since understanding of the ideas included in these exercises is basic to later work with ordered pairs.

---

3 / 32 Exploring ideas / Pages 130-133

Graphs of sets of ordered pairs

Objectives

■ The students learn to make graphs of universes and solution sets of conditions in two variables.

Content overview

■ In this lesson and in the next lesson, we us finite Cartesian sets as universes. In lesson 34 w develop the idea of an infinite Cartesian set, an this set is used as a universe for the variables.

The making of a graph is not an end in itself Nor is this lesson designed to show how to deter mine the solution set of a condition by using graph. One of the most important objectives o this lesson is to help students "picture" the rela tion between a universe and the solution set of condition. The students can more easily "picture" this relation when they use the graph of the uni verse to make a graph of the solution set of th condition. Thus the students see that the graphs of all the solution sets are "embedded" in th graphs of the universe, and from this they should see that every solution set is a subset of the uni verse.

Remember that a graph is a symbol; it is no an idea. A graph of a set of numbers is a set o dots, and each dot represents a point that is asso ciated with a number. Each dot in a graph of a set of ordered pairs, however, represents a poin that is associated with an ordered pair of num bers.

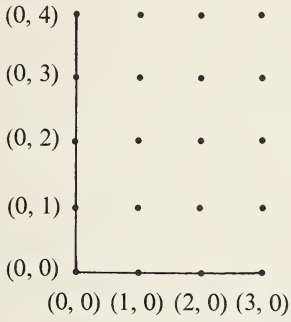
To make a graph of a set of single objects, we use a picture of a straight line. To make a graph of a set of ordered pairs, we use a picture of a plane. The plane is determined by the two num ber lines that are used for the axes.

Notice that, since the axes are number lines (ideas), they are pictured, or represented. The pic ture of an axis is not *the* axis.

Look at D3 on page 131 of *STM 1*. Notice that the points on the first and second axes, such as point F and point G, have a dual purpose. Point F, for example, is associated with the num ber 1 from set A. This tells us that each point located above point F is also associated with the number 1 from set A. Each of these points is one

nit to the right of the second axis. However, oint F is also associated with (1, 0) from A × B.

Point G is associated with 4 from set B and), 4) from A × B. Each point located to the right f point G is four units above the first axis. We ould label the dots on the first and second axes in he way shown below. It is customary, however, o label the dots in the way shown in D3.



Since the points in the axes are also in the coördinate plane, they correspond not only to single members, but also to ordered pairs. The interpretation made depends on the context of the discussion.

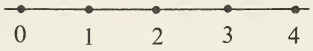
The definition of *graph* in lesson 13, page 52, can now be extended. It can be said that a graph of a set of ordered pairs is a set of dots, each dot of which represents a point associated with a member of the set of ordered pairs. Each member of the set of ordered pairs is associated with a point represented by a dot in the graph.

Notice that we have not mentioned distance in our discussion of the points that are represented in a graph. No measure is involved in this type of graph. There is no distance concept here.

The two explanatory paragraphs that follow exercise D on page 130 describe an arbitrary choice that is usually made by mathematicians.

Instruction notes

■ The following procedure can be used to introduce the lesson. Put the following on the board:



Say: "Back in lesson 13, you learned to make graphs such as the simple graph I have made on the board. Were they graphs of sets of single objects or graphs of sets of ordered pairs of objects?" [Single objects] "In this graph on the board, what is indicated by the numeral 2 and the dot above it?" [That the number 2 is associated with the point. The point is represented by the dot.] "In this lesson you will learn how to make a graph of a set of ordered pairs."

After exercise K on page 131, write on the board "C = {0, 1, 2, 3, 4, 5}; D = {0, 1, 2}." Ask: "How many members are there in C × D?" [18] "How many dots will be in a graph of C × D?" [18] Sketch a graph of C × D. Point to a dot in the graph and ask a student what ordered pair is associated with the corresponding point. Do the same for several other dots.

Whenever possible, point out how the solution set is related to the universe. For example, after exercise M on page 133, ask: "How many graphs are in D8?" [Two] "What are the two graphs?" [One is a graph of A × A, and the other is a graph of the solution set of $y + 3 = x$.]

If possible, duplicate copies of graphs of the universe for "On your own" exercises 10 through 13. If this is not possible, provide students with commercially made graph paper, or have them buy cross-ruled paper. Using duplicated copies of graphs of the universe or cross-ruled paper can reduce the time needed for the exercises.

A "graph board" is a very effective visual aid and learning device. One way to make such a board is to mark off dots on a sheet of plywood. The dots are the graph of the universe. Paint a picture of the first axis and the second axis on the board.

Bore holes at each of the dots on the board and insert dowels to represent the dots. Cut disks from a large dowel and bore holes in the center of each disk so that the disks can be placed on the dowels that have been inserted into the board. The disks can represent the dots that are the graph of a solution set.

The board can also be used for future work to represent graphs of solution sets of compound conditions in two variables. Square-shaped blocks of wood may be cut and used in a manner similar to the way the disks are used. Paint the disks and the square-shaped blocks of wood in different colors.

Provision for individual differences

■ The following developmental exercises might be omitted by a class of abler students:

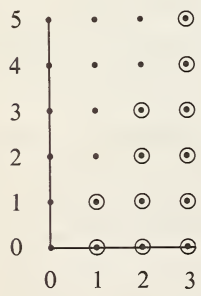
C, page 130

F, H, K, page 131

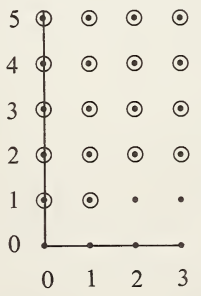
A, B, E, page 132

Abler students might consider the following:

1 Consider $A \times B$ when $A = \{0, 1, 2, 3\}$ and $B = \{0, 1, 2, 3, 4, 5\}$. Use $A \times B$ as the universe for (x, y) in $2x > y$. Make two graphs to show that the graph of the solution set of $2x > y$ is not the same when $A \times B$ is the universe for (x, y) as when $A \times B$ is the universe for (y, x).



Universe for $(x, y) = A \times B$.



Universe for $(y, x) = A \times B$.

$2x > y$.

2 Using $A \times B$, when $A = \{0, 1, 2, 3\}$ and $B = \{0, 1, 2, 3, 4\}$, as the universe for (x, y), tabulate each of the sets named below:

(a) The intersection of the solution set of $x = y$ and the solution set of $x > y$. [{ }]

(b) The union of the solution set of $x = y$ and the solution set of $x > y$. [{(0, 0), (1, 1), (2, 2), (3, 3), (1, 0), (2, 0), (2, 1), (3, 0), (3, 1), (3, 2)}]

(c) The intersection of the solution set of $y + x < 5$ and the solution set of $x > y$. [{(1, 0), (2, 0), (2, 1), (3, 0), (3, 1)}]

(d) The intersection of the solution set of $2x + y = y$ and the solution set of $x + y < 3$. [{(0, 0), (0, 1), (0, 2)}]

Slow learners enjoy and derive a great deal of satisfaction from making graphs if they understand how to make them. You might discuss with these students "On your own" exercises 7, 8, and 9 to clear up any questions they may have on making graphs.

---

3 / 33 Exploring ideas / Pages 134-137

Conditions and graphs

Objectives

■ The students learn to use a graph to find the solution set of a condition in one variable when the universe is a set of ordered pairs. They also learn to obtain a condition from a graph of the solution set of the condition.

Content overview

■ The ideas developed in the preceding lesson are used in this lesson. A set of ordered pairs is used as the universe for a variable in a condition involving just one variable. Under these circumstances, the condition is a requirement for just one of the components of each member of the solution set. For the condition expressed in D2, the

niverse is a set of ordered pairs. The requirement s that the first component of each member of the niverse that satisfies the condition must be 3 to btain a true statement from the condition. There s no requirement for the second components of he members of the solution set. Therefore, each nember of the universe whose first component is is a member of the solution set of $x = 3$, and his set is {(3, 0), (3, 1), (3, 2), (3, 3), (3, 4)}. Similarly, each member of the universe for (x, y) vhose second component is 3 is a member of the olution set of $y = 3$. This set is {(0, 3), (1, 3), 2, 3), (3, 3), (4, 3)}.

On page 136 the idea is developed that a condition can be determined from the graph of the solution set of the condition. Actually, the same graph may be the graph of the solution set of several equivalent conditions. For example, the graph in D8 is a graph of the solution set of $x + 2 = y$, $y - 2 = x$, and $x + 3 = y + 1$. Notice that, even though these three conditions are different from one another, they are equivalent to one another; that is, they all have the same solution set when the universe for the variables is the same.

Instruction notes

■ In the presentation of the lesson you can help students understand that when $U = \{0, 1, 2, \ldots, 10\}$ the solution set of $x = 3$ is not the same as when $A = \{0, 1, 2, 3, 4\}$ and the universe for $(x, y) = A \times A$. (See D1, D2, and D3 on page 134.) One way that you might do this is suggested below.

After exercise N (column 1) on page 135, ask: "Is the condition expressed in D1 the same as the condition expressed in D3?" [Yes] "Have you found that a condition can have more than one solution set?" [Yes] "Suppose that the universe for x in $x = 3$ is $\{0, 1, 2\}$. When this set is the universe, what is the solution set of $x = 3$?" [The empty set] "What determines the solution set of a condition?" [The universe for the variable(s) and the condition together]

For exercises K through N (column 2) on page 135, you may want to make a graph of $R \times R$ on the chalkboard.

After exercise C on page 136, say: "Can the set you tabulated for exercise A be the solution set of $y < x$?" [Yes] "When two conditions have the same solution set, they are sometimes called equivalent conditions."

After exercise F, write these two sentences on the chalkboard: $x + 2 = y$; $y - 2 = x$. Then say: "Is each of these conditions satisfied by the set you tabulated for exercise D?" [Yes] "Can you think of another condition that also is satisfied by this solution set?" [$x + 3 = y + 1$ is an example.]

When you assign the "On your own" exercises, supply your students with graph paper for exercises 1 through 6, if possible.

Provision for individual differences

■ The following developmental exercises could be omitted by a class of abler students:

A, F, I, page 134

A, D, G (column 1), M (column 2), page 135

A class of abler students might write sentences that express two or three conditions that have the same solution set as the solution set they tabulate for each of "On your own" exercises 13 through 17.

For a class of slow learners, you could discuss pages 134 and 135 and assign "On your own" exercises 1 through 12 on the first day. Then you could use page 136 on the second day and make several more graphs of universes and solution sets on the chalkboard. Let the students work out the conditions whose solution sets are indicated by the encircled dots in these graphs. Then assign "On your own" exercises 13 through 17.

Slow learners will have difficulty in finding solution sets of conditions involving one variable when the universe is a set of ordered pairs (for example, $x = 3$, universe for $(x, y) = A \times A$ from D2 and D3 on page 134). Suggest that they think of a second condition that is implied by $x = 3$. The second condition implied is "y can assume any value of A." The condition "y can assume any value of A" should help them concentrate on the second components of ordered pairs that satisfy the condition $x = 3$. This is a "crutch," and it should only be used to help slow learners understand that the solution set of $x = 3$ is related to the universe.

If you wish, you can omit the developmental exercises on page 136 and "On your own" exercises 13 through 17 for slow learners.

---

3 / 34 Exploring ideas / Pages 138-142

The Cartesian set N × N

Objectives

■ The students learn to use a standard description to name the solution set of a condition when the universe for the variables is a set of ordered pairs. Then they learn to make a graph of the Cartesian set formed from the set of natural numbers and, following this, a graph of the solution set of a condition when $U = N \times N$.

Content overview

■ A standard description of a solution set of a condition in two variables is similar in some respects to the standard descriptions of solution sets of conditions in one variable, with which the students are familiar. A standard description always includes a sentence that expresses a condition that the members of the set described must satisfy. Only the members of this set, and no other members of the universe, satisfy the condition.

If A and B are both finite sets, then $A \times B$ is a finite Cartesian set. When set A is finite, a natural number is associated with set A. The natural number is the number of members of A. If B is also a finite set, then a natural number is associated with set B. The number of members in $A \times B$ is the product of the number of members in A and the number of members in B. This holds true when either A or B, or both A and B, are the empty set. Zero is the natural number associated with the empty set. The Cartesian set formed by the empty set and any other set is the empty set, which is a finite set.

In general, any Cartesian set $A \times B$ is an infinite set if one (or both) of the sets A and B is infinite and the other set is not the empty set. If $A = \{5\}$, then the Cartesian set $A \times N$ is a set of ordered pairs in which the first component of each member is 5. The second component of each member is a natural number. Since the natural numbers form an infinite set, the set of ordered pairs having a first component of 5 and a second component that is a natural number is an infinite set. Therefore, $A \times N$ is an infinite set. The Cartesian set formed from the set of natural numbers ($N \times N$) is, of course, an infinite set.

Some infinite sets of ordered pairs cannot be tabulated so that they can be interpreted intelligibly. Except for certain sets that can be used as a universe, for example, $N \times N$, the students will have to rely on a standard description to identify an infinite set. More work with standard descriptions of infinite sets is provided in unit 7.

When the universe is an infinite set, it is impossible to use each ordered pair as a replacement for the variables in a condition for (x, y). In the next lesson we show how charts can be used to help find the solutions of conditions in two variables. In this lesson a knowledge of arith-

netic is relied on to find solutions of simple conitions such as $x = y$ and $x + y = 6$.

Solutions of conditions for equality can be sed to help find the solutions of corresponding onditions for inequality. For example, the soluions of $x + y = 6$ can be used to find the soluions of $x + y > 6$.

Instruction notes

You might start this lesson by reviewing standrd descriptions. Say: "Let us review some ideas ou have studied about standard descriptions. What is described by a standard description?" The solution set of a condition] "Do we always se braces when describing or tabulating a set?" Yes] "What is named first within the braces?" The variable] Write "$\{x$ $\}$" on the chalkoard. "What symbol follows the name of the ariable?" [A vertical stroke that stands for the words "that satisfy the condition that"] Write "$\{x|$ $\}$." "What follows the stroke?" [A sentence that expresses a condition] Write on the board "$\{x|x - 2 > 5\}$." Say: "You are going to use these ideas now to learn how you can write a standard description when the universe is a set of ordered pairs." Begin the developmental exercises.

After exercise C on page 138, say: "In the last lesson you worked with the condition $x = 3$ when the universe was a set of ordered pairs. Think about how you would write a standard description of the solution set of that condition." Write the following standard descriptions on the chalkboard: $\{x|x = 3\}$ and $\{(x, y)|x = 3\}$. Ask: "Which of these descriptions should you use for the solution set of the condition we are talking about?" [$\{(x, y)|x = 3\}$] "Why did you make this choice?" [Because this description indicates that both the universe and the solution set are sets of ordered pairs]

Following exercise F on page 140, tell the students to consider set A, whose members are 0, 1, 2, 3, 4, 5. Ask: "Does D8 show a complete graph of $\{(x, y)|y = x\}$ when the universe for (x, y) is $A \times A$?" [Yes] "Which is this solution set, an infinite or a finite set?" [Finite]

After exercise Q on page 141, ask: "If the universe for (x, y) is $N \times N$, can you make a complete graph of $x + y < 4$?" [Yes] "If the universe for (x, y) is $N \times N$, can you make a complete graph of $x + y > 4$?" [No]

Provision for individual differences

Abler students can complete this lesson on a self-study basis. Discuss the "On your own" exercises with them. Then have them consider under what conditions $A \times B$ is an infinite set [When either A or B, or both A and B, are infinite sets and neither is the empty set]; $A \times B$ is a finite set that is not the empty set [When both A and B are finite, but neither is the empty set]; $A \times B$ is the empty set [When either A or B, or both A and B, are the empty set].

For a class of slow learners, you will want to spend two periods on this lesson. On the first day you could use pages 138 and 139 and assign "On your own" exercises 1 through 9. Then you could finish the preliminary exercises on the second day and assign "On your own" exercises 10 through 19.

3 / 35 Exploring ideas / Pages 143-146

Charts and solution sets

Objectives

The students learn to use a chart to find solutions of a condition in two variables. First they replace one of the variables, and then they determine the replacement for the other variable that makes a true statement from the condition.

Content overview

■ We have shown how computation is used to help find solutions of conditions in one variable. Now we make a reasonable replacement for one of the variables in a condition in two variables. This replacement is one of the components of a member of the solution set. Then we use computation to find the other component of a member of the solution set of the condition. This method can be used to find solutions of conditions when the universe is finite or when the universe is infinite.

It is necessary to decide for which of the two variables in a condition a replacement should be made first. In the condition $y = 5 - x$, discussed on page 143, we replace x so that we obtain a condition in one variable that can be solved by computation. In the condition $x + y = 4$, we may replace either variable because there is no advantage in replacing one rather than the other.

Notice that for the condition $s = 2r + 1$ (page 145), we replace r to obtain a condition in one variable, such as $s = 2(0) + 1$. If a replacement were made for s instead of r, a condition such as $2 = 2r + 1$ would be obtained. We would then have to consider $1 = 2r$. Since $\frac{1}{2}$ is not a member of N, $(\frac{1}{2}, 2)$ is not a solution. For many conditions, replacing one variable rather than the other provides a shorter and easier way to determine the solution set of the condition.

After the variable that is to be replaced first is selected, it is necessary to decide which numbers should be used as replacements for that variable. As a comparison of the universes and the solution sets of the conditions expressed in D1, D2, and D3 on page 143 shows, not every component of a member of the universe is necessarily a component of a solution. To decide which numbers should be used as replacements, a knowledge of arithmetic and an understanding of the requirement made by the condition must be relied upon. For the condition $y = 5 - x$, the replacement for x can be any natural number from 0 through 5. The condition requires that the difference of 5 and each replacement for x be the same member of A as the replacement for y. Since all possible replacements of x from A satisfy this condition, each first component of the universe $A \times A$ can be used as a replacement for x. The condition $x + y = 4$ requires that the sum of the replacement for x and the replacement for y be 4. Hence, each replacement for x that is a component of a member of the solution set is less than 5. Only the natural numbers from 0 through 4 are used as replacements for x.

In the condition $s < 2r + 1$ (page 145), when $U = N \times N$, there is an infinite set of possible replacements for r that are components of solutions of the condition. Just some of these numbers can be used as replacements. Only a few members of the infinite solution set can be found. The set of ordered pairs that are obtained from a chart is a subset of the solution set of the condition. It should be pointed out that an infinite universe does not necessarily mean that the solution set will be infinite. For example, the solution set of $7 - x > y$, when $U = N \times N$, is a finite set.

The use of a chart provides a systematic method of finding the solution set of a condition. As soon as the student has mastered the thought processes involved in finding solutions of conditions in two variables, the making of such charts may be discontinued.

Instruction notes

■ During the presentation of this lesson, you will have an opportunity to review with your students how to find solutions of simple conditions. For example, before beginning exercise C on page 143, write on the chalkboard "$x + 9 = 15$. $U = N$." Ask: "How do you find the solution of this condition?" [By subtracting 9 from 15] Then write on the board "$17 = x - 6$. $U = N$." Ask:

'How do you find the solution of this condi-ion?" [By adding 6 to 17] "You are going to apply what you have learned about finding the solution of a condition in one variable to finding the solutions of a condition in two variables. You are going to see how you can make a chart that will help you find the solutions of a condition in two variables."

After exercise F on page 143, say: "When you replace x by 1 in $y = 5 - x$, what condition do you obtain?" [$y = 5 - 1$.] "What number satisfies this condition?" [4] "Where does the numeral 4 belong in a chart like the one in D2?" [Opposite the letter y and directly below the numeral 1]

Before exercise G on page 145, say: "Think about the set of ordered pairs you obtained from your chart. Is this set a subset of the solution set of $s = 2r + 1$?" [Yes] "Is it possible to tabulate the solution set of $s = 2r + 1$?" [No] "Can you identify this solution set by a standard description?" [Yes] "Why would you use a chart for this condition if you are not going to tabulate the solution set?" [To obtain some of the members of the solution set; if this condition were one of the simple conditions of a compound condition, you would have to find some of the solutions to find the solutions of the compound condition.]

Provision for individual differences

■ A class of abler students would not need to tabulate $A \times A$ for exercise A on page 143. The following developmental exercises also could be omitted by abler students:

B, E, H, page 143

M, Q, I, page 144

With a class of abler students you could present this lesson by means of class discussion. Copy each of the displays on a clear sheet of acetate and use the resulting transparencies with an overhead projector, or make copies of the displays on the chalkboard. Refer to these copies of the displays and use a sequence of questions similar to those in the developmental exercises.

With a class of slow learners, use a methodical and deliberate approach. If possible, provide duplicated copies of D2 and D3 that have not been filled in. Have the students fill them in as you lead a discussion on the use of the charts. The students could also make charts on ruled paper. Using class time for "making charts" should be avoided, if possible. The students should concentrate on using charts to find solutions of conditions.

If you use two periods for this lesson, on the first day stop with exercise Q, page 144, and "On your own" exercise 8. If possible, give students help in completing the first few "On your own" exercises in class.

---

3 / 36 Exploring ideas / Pages 147-150

Compound conditions in two variables

Objectives

■ The students learn to use graphs and charts to help them find the solution sets of compound conditions in two variables.

Content overview

■ Since the solution set of a compound condition that involves the connective "and" is the intersection of the solution sets of the simple conditions, the graph of the solution set of the compound condition contains those dots that are in the graphs of both simple conditions. Look at D2 on page 147. Notice that four graphs are pictured: a graph of $N \times N$, a graph of $\{(x, y) | x + y = 6\}$, a graph of $\{(x, y) | y = 2x\}$, and a graph of $\{(x, y) | x + y = 6 \wedge y = 2x\}$.

A chart can be used to find the solution set of a compound condition by first obtaining the or-

dered pairs that satisfy one simple condition and then selecting from among these the ordered pairs that satisfy the other simple condition. The charts below show how the chart in D4, on page 148, when completed, may be used to find the solution set of the condition expressed in D3, $x + y < 7 \wedge y > x$. The universe for (x, y) is $N \times N$. Since the solution set of $x + y < 7$ is a finite set, the chart will be used first to find this solution set.

$x + y < 7.$

x	0	1	2	3	4	5	6
y	0	0	0	0	0	0	0
	1	1	1	1	1	1	
	2	2	2	2	2		
	3	3	3	3			
	4	4	4				
	5	5					
	6						

From the chart, we can obtain the ordered pairs that satisfy $y > x$. It is obvious that (0, 0), (1, 0), (1, 1), and so on, do not satisfy $y > x$. Therefore, we can cross out the numerals for the second components that are not greater than the first components. The chart then looks like this:

x	0	1	2	3	4	5	6
y	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~
	1	~~1~~	~~1~~	~~1~~	~~1~~	~~1~~	
	2	2	~~2~~	~~2~~	~~2~~		
	3	3	3	~~3~~			
	4	4	4				
	5	5					
	6						

The twelve ordered pairs indicated by the numerals that are not crossed out form the solution set of $x + y < 7 \wedge y > x$.

We extend the work with graphs and charts to find solution sets of compound conditions for problems in lessons 38, 39, and 40.

Instruction notes

■ Ideas studied in previous lessons are utilized in this lesson, and students should, therefore, find this lesson rather easy. Be sure to emphasize the important ideas. For example, after exercise H on page 147, ask: "How can you tell by just looking at the graph in D2 which dot is in the graph of the solution set of the compound condition?" [By noting the dot that is both encircled and enclosed by a square]

After the first question in exercise O on page 148 has been answered, say: "You know that the ordered pair whose first and second components are both zero is indicated in your chart by the numeral zero in the x row and the numeral zero under it in the y row. Does this ordered pair satisfy $y > x$?" [No] "Since it doesn't, you can cross off the numeral zero in the y row. Do the rest of the ordered pairs indicated in this column satisfy $y > x$?" [Yes] "Look under the 1 in the x row. What ordered pairs indicated in this column do not satisfy $y > x$?" [(1, 0) and (1, 1)] "Cross off the numerals 0 and 1. Cross off the other numerals on your chart that do not indicate ordered pairs that satisfy $y > x$." Discuss with the students how the chart was helpful in finding the ordered pairs that satisfy the compound condition.

Finish the lesson. Then say: "Let us review some ideas you have learned in this lesson. Suppose that you are going to make a chart or a graph to help you find the solution set of a compound condition. Which simple condition should you start with?" [You should start with the simple condition that has a finite solution set.]

You could present the entire lesson by using class discussion to develop the ideas. If you do this, have students keep their books closed. Reproduce the displays on the chalkboard, use

sketches on sheets of acetate and an overhead projector, or provide students with duplicated copies of the displays. Do not complete the graphs, but have the students do this during the class discussion.

If you use three sheets of clear acetate on which to reproduce D2 on page 147 and D6 on page 148, you can emphasize the meaning of "intersection" in the following way. First, show a graph of the universe. Second, show a graph of one of the simple conditions. Then show a graph of the second simple condition. If your "overlays" are carefully made, the intersection should be very evident to students.

Provision for individual differences

■ Abler students will find this an interesting lesson. You could have them study the developmental exercises independently. Then you could follow through with a thorough discussion of the "On your own" exercises.

As we said before, slow learners enjoy making graphs, but for these students the lesson may prove to be too long for one class period. Plan to use two days on this lesson with slow learners, completing developmental exercises A through Q on pages 147 and 148 and "On your own" exercises 1 through 4 on the first day. Utilize whatever visual aids you may have for this material. Provide students with duplicated copies of graphs of the universe or with cross-ruled paper to reduce the amount of time needed for making graphs.

3 / 37 Exploring problems / Pages 151-153

Simple conditions for problems

Objectives

■ The students learn to find answers to problems that involve simple conditions in two variables.

Content overview

■ As has been noted before, one of the major goals of the *STM* program is to develop skill in problem solving. We think that after a mathematical idea has been developed it should be applied to solving problems. The mathematics used to solve problems in unit 1 involved a simple condition in one variable. In unit 2 compound conditions in one variable were used. In this lesson simple conditions in two variables are used. In lessons 38, 39, and 40, compound conditions in two variables are used for problems. In unit 4 conditions for equivalence are introduced. These are also used to help solve problems. Part of unit 7 is also devoted to problem solving.

When a problem contains two undetermined quantities, a condition for that problem usually involves two variables. For the problem in D1 on page 151, both the number of fish that George could have caught and the number of fish that Steve could have caught are not known. A different symbol is used for each of these variables. Two variables are used because there are two different undetermined quantities.

Whether the action indicated in a problem is real or imagined, it should be reflected in a condition for that problem. For the problem in D1, we can imagine that the group of fish Steve caught is put with the group of fish that George caught. Since we imagine combining one group with another group, we can think of the action as additive. Because the two boys together caught fewer than 5 fish, the sum of the replacements for x and y is less than 5.

Notice that the answer to this problem cannot be "Steve caught 0, 1, 2, 3, or 4 fish and George caught 0, 1, 2, 3, or 4 fish." This answer is not correct because it implies, for example, that the boys could each have caught 4 fish. The answers given after exercise J on page 151 are precise.

The events described in D3 on page 152 are, in sequence, as follows: Judy had 8 stamps; she used some of them, but had some left. Our thinking can be: As Judy used the stamps, she removed them from the original 8 stamps; when she stopped using the stamps, she had some left. If we reconstruct the events in the order described, we obtain the condition $8 - m = n$. Other conditions are equivalent to $8 - m = n$, for example, $m + n = 8$, and the correct answers to the problem can be obtained by using $m + n = 8$. But this condition does not represent a direct analysis of what happened. $m + n$ indicates that Judy put the stamps she used with the stamps she had left.

Notice that the universe for some ordered pairs of variables is $N \times N$, while for others the universe is $C \times C$. For the problem in D3, $C \times C$ is used as the universe for (m, n) to eliminate the possible event that Judy used 0 stamps. Also, the condition $8 - m = n$, when the universe for (m, n) is $C \times C$, has the same solutions as the compound condition $8 - m = n \wedge m \neq 0 \wedge n \neq 0$ when the universe for (m, n) is $N \times N$. Thus, by using $C \times C$ as the universe, we have eliminated the need of using compound conditions for certain problems in this lesson.

Instruction notes

■ Before beginning this lesson, you might remind the students of some of the material previously covered. Say: "In units 1 and 2 you learned how to use both simple and compound conditions in one variable to help you get answers for problems. This lesson will help you find the answers to other kinds of problems."

After exercise K on page 151, you might ask: "What does it mean to verify an answer?" [You check to see if the answer agrees with the information given and if it answers the question asked in the problem.]

Point out that $m + n = 8$ should not be used to express a condition for the problem in D3. After exercise M on page 152, write on the chalkboard "$m + n = 8$. The universe for (m, n) is $C \times C$." Say: "Think again about the problem in D3. Is the set you obtained from your chart also the solution set of this condition?" [Yes] "What does this condition indicate?" [That Judy put the stamps she used with the stamps she had left] "Do the sentences in D3 tell you that Judy put the stamps she used with those she had left?" [No] "The reason you cannot use $m + n = 8$ as a condition for the problem is because it does not describe the events in the problem."

Provision for individual differences

■ "Exploring problems" lessons 37 through 40 are closely related. A class of abler students might be able to cover all of them in about three days' work. Notice the "Keeping skillful" exercises on page 156, the "Special challenge" exercises on page 160, and the "Applying mathematics" problems on pages 162 and 163 that you can use to provide enrichment and additional practice. Problems 1, 2, 3, 5, 11, and 13 of "Applying mathematics" can be used with lesson 37.

Slow learners may experience considerable difficulty with problem solving. Give them individual help in making charts for the "On your own" problems. It is important that they clearly see how the charts and graphs can be used to find answers to problems. If many students have severe reading handicaps, it might be helpful to have different students read the problems aloud.

---

3 / 38 Exploring problems / Pages 153-156

Compound conditions for problems

Objectives

■ The students learn to solve problems involving compound conditions in which one simple con-

ition has one variable and the other simple condition has two variables.

Content overview

For each of the problems in this lesson there re two quantities to find. Thus, two variables are nvolved. Because each problem involves two requirements, the condition for the problem is a ompound condition. One of the simple conditions involves only one variable; therefore, it is ossible to find the solutions of that condition ithout using a chart or a graph. The solutions of he condition in one variable can be used as replacements for that variable in the other simple ondition. Lessons 39 and 40 concern problems hat involve compound conditions in which each imple condition has two variables.

For the problem in D1 on page 153, the solution of $r + 3 = 12$ is the first component of each olution of the compound condition. This solution, 9, is then used as a replacement for r in $+ s = 22$ to obtain $9 + s = 22$. The replacement or s that satisfies $9 + s = 22$ is the second component of the solution of the compound condition because the universe is for (r, s). Thus, the olution set of the compound condition is $(9, 13)\}$.

Notice that, even though the compound condition for the problem in D3 on page 154 involves wo variables, only the second components of the solutions are used to obtain answers to the problem. When the students verify their answers to a problem that involves a compound condition, they must be certain that each answer satisfies both of the requirements in the problem.

It is always important to keep in mind that when the condition for a problem involves two variables the universe is for an ordered pair of variables. For the problem in D1 on page 153, if the universe is for (r, s), and if r is used as a variable for the number of records Sue has, then the first component of the solution of the condition is the number of records Sue has. On the other hand, if the universe is for (s, r), and if r is used as a variable for the number of records Sue has, then the second component of the solution is the number of records Sue has.

Instruction notes

■ Discuss the following ideas as you present the lesson. Before exercise I on page 153 is answered, say: "Think about the compound condition expressed in exercise G. For which of the simple conditions is it easier to find the solution?" [$r + 3 = 12$.] "Is there only one replacement that satisfies $r + 3 = 12$?" [Yes]

After exercise N on page 154, say: "Think about verifying your answers to a problem that involves a compound condition. How many requirements must each answer satisfy?" [Two]

After exercise G on page 155, remind students that it is a good idea to make a chart to help find the solutions of a condition. Make the following chart on the chalkboard:

w	13
z	

Say: "Should I write the numeral zero in the row opposite the letter z?" [No. Zero is not a member of C.] "What numerals should I write in the row opposite the letter z?" [1 through 10] "Remember that a chart will help keep you from overlooking any ordered pairs that may be solutions when you tabulate a solution set."

Provision for individual differences

■ For a class of abler students, you could also assign problems 4, 6, 7, 12, 14, and 15 from "Applying mathematics" on pages 162 and 163 in addition to the "On your own" problems on pages 155 and 156.

If slow learners have difficulty in finding the solutions of a compound condition, help them make a chart similar to the one shown in the "Instruction notes" for this lesson. Be careful to avoid a lengthy assignment that will discourage slow learners.

3 / 39 Exploring problems / Pages 156-159

Compound conditions for more complex problems

Objectives

■ The students learn to solve problems that involve compound conditions in which each simple condition has two variables.

Content overview

■ For each problem in this lesson, the solution set of one of the simple conditions is a finite set, but the solution set of the other simple condition is an infinite set. The solution set of the compound condition is therefore a finite set.

We now apply the ideas presented in lesson 36 to finding the solutions of a compound condition for a problem. After we determine the condition for a problem, we use a chart to help find the solutions of the simple condition that has a finite solution set. Then we decide which solutions named in the chart also satisfy the other simple condition. The method described is a powerful tool that can be used to solve problems in which the answers are not at all obvious.

Notice that, for the problem in D1, $y + x$ could be used instead of $x + y$. For the problem in D3, $n + m$ could be used instead of $m + n$. For the problem in D5, $b + a$ could be used instead of $a + b$. In each of these three problems, the action of combining two groups is imagined. The groups can, therefore, be combined in either order. These three situations are different from the type of problem in which a man had $9 and then was given $3 more. The latter situation should be interpreted as $9 + 3$, rather than as $3 + 9$, because the events actually occurred in the order suggested by $9 + 3$.

Instruction notes

■ Remind students that in our problem-solving approach we continue to emphasize that the sentence that expresses a condition for a problem should reflect the sequence of events described in the problem.

After exercise O on page 157, say: "Suppose that we use y as a variable for the number of trees Earl planted and x as a variable for the number of trees Earl's father planted. The universe is still for (x, y). What is the condition for the problem now?" [$y + x = 7 \wedge y + 3 = x$ or $x + y = 7 \wedge y + 3 = x$.] "What is the solution set of this condition?" [{(5, 2)}] "Is the same answer obtained to the problem from this solution set?" [Yes]

Provision for individual differences

■ The following developmental exercises could be omitted by a class of abler students:

C, F, page 156
K, C, page 157
A through N, pages 158 and 159

If you choose to omit exercises A through N, listed above, you can develop the condition for the problem in D5 by class discussion; then find the solutions of the condition and obtain the answers to the problem.

For your abler students, you could also assign problems 8, 9, 10, 18, 19, and 20 from "Applying mathematics" on page 163 in addition to the "On your own" problems on page 159. Suggest that they may not need to make a chart to find the solutions of some conditions.

If your class is not able to maintain the time schedule you have planned, or if you have a class of slow learners, you can omit this lesson, since it

s not absolutely essential that the students under-
stand the particular ideas developed in this lesson
o understand more important mathematical
deas presented in succeeding lessons. If, how-
ever, you teach this lesson to slow learners,
emphasize the use of a chart in finding the solu-
ions of a compound condition in two variables,
nd give the students individual help with the
"On your own" problems.

3 / Special challenge / Page 160

Objectives

■ The students generalize intuitively the idea of a power set of a set. Then they make a relatively rigorous application of the idea of intersection of sets.

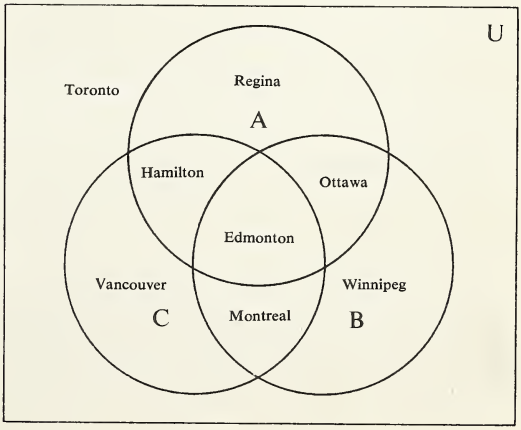
Content overview

■ Exercises A through E concern the idea of a *power set* of a set. A power set of a given set is the set of all subsets of the given set. The power set of {x, y, z} consists of { }, {x}, {y}, {z}, {x, y}, {x, z}, {y, z}, and {x, y, z}. Notice that each member of a power set is a set.

If a set has no members, its power set has 2^0, or 1, member. If a set has 1 member, its power set has 2^1, or 2, members. If a set has 2 members, its power set has 2^2, or 4, members. If a set has n members, its power set has 2^n members. For further reading on this idea, refer to *Introduction to Finite Mathematics* by Kemeny, Snell, and Thompson, as listed in the bibliography in this *Teaching Guide*.

The binary numeration system, whose properties are developed in lesson 51, can also be used to help answer exercise E. For example, consider set S when S = {a, b, c, d}. Think of the members of set S in the order they are listed. For any given subset of S, each member of S is either in the subset or not in the subset. Use the symbol 0 to represent the fact that a unique member of S is not in the given subset, and use the symbol 1 to represent the fact that a unique member of S is in the given subset. Then, for set S, the symbol 1010 indicates the subset {a, c}. Use the binary numeration system to list every possible combination of the members of S. The symbol 0000 indicates { }, 0001 indicates {d}, 0010 indicates {c}, 0011 indicates {c, d}, and so on up to 1111, which indicates {a, b, c, d}. This procedure generates a list of the binary numerals from 0 through 1111 (which in the decimal system is from 0 through 15). Since there are sixteen numerals from 0 through 15, there are sixteen subsets of S.

For exercise F, the Venn diagram below represents the intersection of the three sets. A team could not have played any other team in the same set. Since Edmonton is in all three sets, Edmonton must have played Toronto. Since Ottawa is in both sets A and B, Ottawa must have played Vancouver, which is the only member of C not in either of sets A or B. By the same token, Regina played Montreal and Hamilton played Winnipeg.



For exercise G, you can refer to the back of this *Teaching Guide* to see if a student's sketch is correct. The sketch for this problem is found in the "Responses to Special Challenges."

You might ask abler students to devise a way to show that each line must contain exactly three of the six points. Let the numerals 1, 2, 3, and 4 represent the lines and the letters G, H, I, J, K, and L the six points. There are exactly six possible pairings of the lines whose intersections are the points $G_{1,2}$, $H_{1,3}$, $I_{1,4}$, $J_{2,3}$, $K_{2,4}$, and $L_{3,4}$. $G_{1,2}$ is the point G that is the intersection of lines 1 and 2; $H_{1,3}$ is the point H that is the intersection of lines 1 and 3; etc. It can be seen that each of the four lines contains three of the six points. For example, points G, H, and I are contained in line 1; points G, J, and K are contained in line 2; etc.

Instruction notes

■ You may wish to advise the students not to try to respond to exercises A through E until they have completed lesson 51. However, the students could now answer exercises A through E if you give them the following clue after they complete exercise B: A set with two members has 2×2, or 4, subsets. From the example given before exercise A, it is known that a set with three members has 8 subsets. Also, it is known that a set with three members has $2 \times 2 \times 2$ subsets. This information can be used to predict the total number of possible subsets that a set with a given number of members has.

After the students have developed a Venn diagram for exercise F, you might have them tabulate certain sets such as $A \cap B$, $B \cap C$, $A \cap B \cap C$, and so on.

When the students have completed exercise G (they probably will use a trial-and-error method), you might give them an explanation of a response based upon our explanation in the "Content overview."

Conditions for abstract problems

Objectives

■ The students learn to solve abstract problems.

Content overview

■ The problems in this lesson are concerned with numbers only. As such, the problems are not related to physical situations. All the problems involve conditions in two variables. Some of the conditions for the problems are simple conditions; some are compound conditions with one variable in one of the simple conditions; and some are compound conditions with two variables in each of the simple conditions.

As in the last five lessons, we may find the solutions of each condition with the aid of either a chart or a graph. However, the answers to many of the abstract problems can be obtained by making replacements for the variables in the condition for the problem.

For a "story problem," we obtain answers by using the members of the solution set of the condition for the problem. Since the answer for an abstract problem is a number (or numbers), the answer is given by the tabulation of the solution set of the condition.

Instruction notes

■ The following questions may be helpful as you teach the lesson. Before exercise B on page 160, say: "Think again about the problem in D1. Suppose that, instead of using x as a variable for the first number and y as a variable for the second, you use x as a variable for either of the numbers and y as a variable for the other. Give another condition that you could use for this problem." [$y + x = 7$.]

After exercise C on page 160, ask: "Does $+ y = 7$ have the same solution set as $y + x =$?" [Yes]

After exercise N on page 161, say: "Think gain about the requirement in $x < y$. Does the quirement mean that you cannot replace x by and y by 4?" [No] "Is the requirement that, in rder to obtain a true statement from the condi- on, the replacement for x must be less than the placement for y?" [Yes]

Provision for individual differences

For a class of abler students, you could teach his lesson by using only the displays and class iscussion. You could also assign to these stu- ents problems 21 through 25 from "Applying athematics" on page 163 in addition to the "On our own" problems.

The necessary ideas for dealing with abstract roblems were developed in lesson 21. Therefore, your class is not able to maintain the time chedule you have planned, or if you have a class f slow learners, you can omit this lesson. It is ot absolutely essential that the students under- tand the particular ideas developed in this lesson o understand more important mathematical deas presented in succeeding lessons.

/ 41 Exploring ideas / Pages 167-172

Equivalent ordered pairs

Objectives

The students learn that ordered pairs may be used to represent rates. Then they learn how the definition of equivalent ordered pairs is developed and how the definition is used to decide if two ordered pairs are equivalent. They also learn to find ordered pairs that are equivalent to a given ordered pair.

Content overview

■ In this unit we extend the work with conditions in two variables to include conditions that involve rate pairs. Rate pairs are ordered pairs that are members of a special kind of relation. This relation is a proportional relation, that is, a set of ordered pairs in which each member is equivalent to each of the other members.

Unit 4 contains many different kinds of problems involving rate pairs, including the familiar ratio and proportion problems. We use the definition of equivalent ordered pairs to solve these problems.

You will observe in this unit many of the traditional problems that involve business and per cent. Definitions of commonly accepted business terms are included in the problems, all of which are handled within the general framework of rate pairs.

You will notice that the *STM* program is not structured around social applications of arithmetic, but that applications are brought in whenever they are appropriate. For example, we do not include a lengthy discussion on the social applications of discount. We simply describe the term when the first problem involving discount is presented.

In lesson 41 we focus attention on the proportional relation. In the development of the definition of a proportional relation, we first define what we mean by equivalent ordered pairs. Two ordered pairs, (a, b) and (c, d), are equivalent if and only if $ad = cb$. The universe for the variables is C.

We show that equivalent ordered pairs arise from problem situations that involve rates and comparisons. Thus, we motivate the need for studying equivalent ordered pairs. These ordered pairs are useful in solving many different kinds of problems.

A proportional relation is defined as a set of ordered pairs in which each member is equivalent to each of the other members. This means that a proportional relation is a set of equivalent ordered pairs. The members of a proportional relation are rate pairs. Rate pair terminology is introduced in lesson 42. Rate pairs are used in solving rate and comparison problems.

The following ideas are developed in lesson 41: Ordered pairs may be used to represent rates. Two ordered pairs are equivalent when they possess the "cross-product" property; that is, if the product of the first component of the first pair and the second component of the second pair is equal to the product of the first component of the second pair and the second component of the first pair, then the two ordered pairs are equivalent. A set of ordered pairs is associated with each rate or comparison problem situation, and this set is a proportional relation because it consists of ordered pairs each of which is equivalent to each of the others. Another member of a proportional relation can be obtained from a given member by multiplying (or dividing) both components of the given member by the same counting number. Observe this property: "If each component of (a, b) is multiplied by k, when k is any counting number, then (a, b) is equivalent to (ka, kb)." This property is not introduced at this time, but it can be proved very quickly on the basis of the definition of equivalent ordered pairs. Observe that the two cross products are equal. Therefore, the ordered pairs are equivalent by definition.

In lesson 42, ratios are introduced as names of rate pairs. In *STM 1*, ratios are written with a diagonal bar to distinguish them from fraction numerals. (Fraction numerals are written with a horizontal bar when they are introduced in unit 6.) The notation has been changed from the ordinary notation for ordered pairs to emphasize that a rate pair is a special kind of ordered pair.

Notice the careful distinction that is maintained between a name and its referent. Remember that a referent is the idea expressed by a symbol or a name. A ratio is a symbol or a name. Its referent is a rate pair. Rate pairs are ordered pairs of numbers that are used in a special way. Thus you will observe in *STM 1* that we talk about writing or reading ratios, but we never talk about writing a rate pair.

Observe also that rate pairs are not fractions. The distinction between rate pairs and fractions is extremely important. Rate pairs and fractions have different interpretations in physical situations. A fraction refers to a physical situation in which a whole object has been divided into a certain number of equal parts. A rate pair refers to a physical situation in which rates and comparisons (correspondences between quantities) are involved. There is also a distinction between rate pairs and fractions on the mathematical level. Operations on fractions (or rational numbers) are defined in the usual manner. Operations on rate pairs usually are not defined at the elementary level, but if we choose to define addition and multiplication on rate pairs, the definitions are comparable to vector addition and multiplication. The distinction between rate pairs and fractions is further discussed in the lesson notes for unit 6.

Lessons 41, 42, and 43 are devoted to the development of mathematical ideas, but it should be observed that problems from physical situations are used to motivate the need for mathematical ideas. For example, problems are used to show that rate pairs are necessary to solve certain kinds of problems, but a great deal of attention is given to developing the meaning of a rate pair apart from the problem situation from which it arose. After the mathematical ideas have been developed in lessons 41, 42, and 43, the ideas are again applied to problem situations in lesson 44.

is follows our general procedure, which is to troduce a concept in a physical situation, to end some time developing the understanding of e concept, and then to apply the concept to lving problems.

Lesson 44, the first problem-solving lesson in is unit, involves the solving of rate and comrison problems by using conditions for equivace. In a rate problem the components of the dered pair refer to different kinds of objects. In comparison problem the components of the ored pair refer to the same kind of object. Thus, distinction can be made between a rate problem d a comparison problem. However, the simirity between these two types of problems is uch more powerful than their difference, and we ink it is better to emphasize the similarity rather an the difference. We can talk about a rate of miles per 1 hour and a comparison of $15 to 10, but when (4, 1) and (15, 10) are involved in a ondition or a mathematical statement, they are oth rate pairs.

A definition of per cent is developed in lesn 45. In *STM 1*, per cent problems are handled ithin the general framework of other compari-n problems. A per cent is defined as a rate pair at represents a comparison and has a second omponent of 100. The usual "three cases of per ent" are easily handled within this framework. he lesson notes for lessons 45 and 46 contain a ore detailed discussion of our method of workg with per cent.

Instruction notes

Follow the developmental exercises closely to nsure that students achieve an understanding of he ideas discussed in the paragraphs that follow.

The students should understand how ordered airs are used to represent rates. "3 pencils are old for 12 cents" is an example of a physical ituation that involves a rate. From this situation hey can learn to derive ordered pairs of numbers. The first components of the ordered pairs will be numbers that refer to pencils; the second components will be numbers that refer to cents. (3, 12) can be used to represent the rate at which the pencils are sold.

The students should also understand that many ordered pairs can represent the same rate. Certain pencils are sold at a rate of 3 pencils per 12 cents. Then the pencils are regrouped into 3 equal groups of pencils and cents. By considering the equal groups, the students should see that 3 pencils per 12 cents is the same rate as 1 pencil per 4 cents and 2 pencils per 8 cents. Thus the students obtain ordered pairs that represent the same rate by visualizing the physical situation. They can then tabulate a set of ordered pairs each member of which represents the same rate.

It is also important for students to understand that (1, 4) represents the rate at which the pencils are sold, regardless of how many pencils are sold. This is shown by regrouping the pencils and cents. To emphasize this, you could have the students regroup physical objects, for example, 10 pencils and 40 cents, to show that (10, 40) represents the same rate as (5, 20). In this case they would regroup the 10 pencils and 40 cents into 2 equal groups. Make sure that the students are consistent in showing what the components of the pairs they use refer to. If (3, 12), and not (12, 3), is used to represent the rate at which the pencils are sold, the students should use the numbers referring to pencils as the first components and the numbers referring to cents as the second components of the ordered pairs that represent the rate at which the pencils are sold.

During a class discussion you might have the students tell how much they paid for an item they bought. Have them write names of number pairs on the chalkboard. The first components could be the number of items; the second components

could be the number of dollars or cents paid for the items. Explain that each of the pairs named on the board represents a rate. Since rate does not only apply to prices, you might bring up other examples of rate, such as rate of reading, rate of population growth, rate of absenteeism in a school, and so on.

If the students in your class have had experience with the *STA* program in the elementary grades, you will find that they have an excellent background for the work on rate pairs in this unit.

Provision for individual differences

■ Your abler students, especially those who have had previous experience with the *STA* program, will be able to read and study independently many of the developmental exercises in this unit. You can then talk over the main ideas of each lesson in a general class discussion. Some of the "On your own" exercises can be handled on an oral basis, for example, exercises 1 through 4, 20, and 21 in this lesson.

You can assign the following exercises for additional study for your abler students:

1 The universe for a and b is C.

(a) Is $(a, b) \sim (a, b)$? [Yes]

(b) Is $(a, a) \sim (b, b)$? [Yes]

(c) Is $(a, b) \sim (b, a)$? [No; only when a and b are each replaced by the same number]

(d) Is $(a, b) \sim (2a, 2b)$? [Yes]

(e) Is $(a, b) \sim (a + 2, b + 2)$? [No; only when a and b are replaced by the same number]

2 For each "No" answer given in exercise 1, give an example in which a false statement is obtained.

3 Suppose that you obtain a true statement from $(a, b) \sim (c, d)$. U = C. From which of the conditions expressed below will you also always obtain a true statement? [b, c, e]

(a) $(b, a) \sim (c, d)$.

(b) $(b, a) \sim (d, c)$.

(c) $(a, c) \sim (b, d)$.

(d) $(a, d) \sim (c, b)$.

(e) $(c, a) \sim (d, b)$.

4 When can you obtain a true statement fro $(a, b) \sim (c, d)$ if you know that:

(a) $a > b$ and $c < d$? [Never]

(b) $a = b$? [When $c = d$]

(c) $b = d$? [When $a = c$]

(d) $a = 2b$ and $c = 2d$? [Always]

5 For each replacement of the variables by mem bers of C, is $\{(a, b), (2a, 2b), (3a, 3b)\}$ a propor tional relation? [Yes]

With a class of slow learners, you should pla to spend at least two days on this lesson. On th first day, you might stop on page 169 after th paragraph that introduces the symbol for equiva lence and assign "On your own" exercises through 13. Discuss a few of the "On your own" exercises with the students before they begin thei work. Emphasize the definition of equivalent or dered pairs after students have become familia with its meaning in a physical situation. Mak good use of the displays to assure that student understand the ideas in this lesson before they proceed with the lessons that follow in the unit.

Rate pairs and ratios

Objectives

■ The students learn what a rate pair is and wha a ratio is. They learn how to tabulate an infinite proportional relation and how to write a standard description of an infinite proportional relation. Then they learn to make a graph of an infinite proportional relation.

Content overview

■ In this lesson we define a rate pair as a member of a proportional relation. We want to think about the ordered pairs that are members of proportional relations in a special way. A rate pair is

erefore, a mathematical concept, just as an or-red pair is a mathematical concept.

Many different symbols can be used to express rate pair. For example, 1:2, $\frac{1}{2}$, (1, 2), and 1/2 n all be used, but we have chosen to use a sym-ol like 1/2. A symbol that expresses a rate pair a *ratio*.

We carefully maintain the distinction between symbol and what the symbol represents. Just as ere are names for numbers, there are names for te pairs. Symbols that express numbers are nu-erals, and symbols that express rate pairs are tios.

We do not wish to use the symbol $\frac{1}{2}$ to express e rate of one to two because we use this nota-on for fraction numerals in unit 7. Rate pairs e not fractions, and while the distinction can sily be made in a problem situation, confusion ight occur when the symbol is viewed apart om the problem situation. The use of a diagonal ar in writing a ratio emphasizes to the student at the symbol represents a special kind of or-ered pair, a rate pair.

Previously we used a rate situation to obtain te pairs that form a proportional relation. Now e tabulate an infinite proportional relation such s {4/7, . . .}. Only one member is named in this bulation because, given one member, it is pos-ble to obtain as many members as desired. bviously, not all the members can be named.

Standard descriptions of proportional rela-ons are introduced in this lesson. We use x/y as pair of variables for rate pairs that are equiva-nt to 4/7. The sentence $x/y \sim 4/7$ is a group of ymbols that expresses a condition for equiva-nce. The condition $x/y \sim 4/7$ is the idea ex-ressed by the symbols. The solution set of the ondition is {4/7, . . .}. Hence, the solution set is proportional relation, and $\{(x, y) | x/y \sim 4/7\}$, hen $U = C \times C$, is the same set as {4/7, . . .}. Thus, an understanding of a standard description of an infinite proportional relation is based on an understanding of a condition for equivalence and a standard description of the solution set of a simple condition in two variables.

Instruction notes

■ After exercise B on page 173, say: "Turn back to D7 on page 170. Are the members of set B rate pairs?" [Yes] "Name two other sets whose members are rate pairs." [For example, {(1, 3), (2, 6)} and {(2, 5), (4, 10), (6, 15)}] "Are the sets you just named proportional relations?" [Yes]

After exercise G on page 173, have a student read the tabulation in D1. [The set whose members are 1 to 15, 2 to 30, 3 to 45, 4 to 60, 5 to 75]

Before discussing the exercises in column 2 on page 174, ask: "How many members are in the set named in D1 on page 173?" [5] "How do you know that this is a finite set?" [Because the number of members in the set is a natural number] "Have all the proportional relations you have studied so far been finite sets?" [Yes] "Now you will learn about proportional relations that are infinite sets."

After exercise I on page 175, ask: "Besides using a tabulation, in what way can we name a set?" [By using a standard description] "So far you have used only a tabulation to name a proportional relation. Now you will learn to use a standard description."

For the sets named in exercises N through Q on page 175, have your students name three members of each set. [For exercise Q, accept either m/n, n/m, $2n/2m$, $3m/3n$, etc., or 1/1, 2/2, 3/3, etc.]

Provision for individual differences

■ Have your abler students work the following exercises:

1 Tabulate $\{(x, y) | x/y \sim 2/3 \wedge y = 9\}$. $U = C \times C$. [{6/9}]

2 Tabulate $\{(x, y) | x/y \sim 2/3 \wedge x = 9\}$.
$U = C \times C$. [{ }]

3 Tabulate $\{2/3, \ldots\} \cap \{3/2, \ldots\}$. [{ }]

4 Tabulate $\{(x, y) | x/y \sim 5/4 \wedge y > x\}$.
$U = C \times C$. [{ }]

5 Tabulate $\{(x, y) | x/y \sim x/2 \wedge y/x \sim y/3\}$.
$U = C \times C$. [{3/2}]

If you devote two days to presenting this lesson to a class of slow learners, stop with exercise I, page 175, on the first day and have the students work "On your own" exercises 1 through 12. Finish the lesson on the next day.

---

4 / 43 Exploring ideas / Pages 176-179

Solution sets of conditions for equivalence

Objectives

■ The students learn to find solutions of conditions for equivalence.

Content overview

■ The development presented is a necessary prelude to solving rate and comparison problems. Although this lesson is not concerned with problem solving, we have used a problem situation (D1 on page 176) to introduce the work with conditions for equivalence. In lesson 44 the students will learn how to use conditions for equivalence to solve rate problems.

The first condition that is considered in the lesson is $16/8 \sim 4/x$. It requires one to find the replacement for x that makes $4/x$ equivalent to $16/8$. By the definition of equivalent ordered pairs, the condition $16x = 32$ is obtained from the condition $16/8 \sim 4/x$. Thus, the number that satisfies $16x = 32$ also satisfies $16/8 \sim 4/x$. At this level, the students can find the solution of $16x = 32$ by visualizing 16 equal groups. They can imagine that, if they put one marker in eac of 16 groups and then repeat this process unt they have used 32 markers, there will be tw markers in each of the 16 groups.

From situations like this, the students shoul see that, in general, the replacement for b tha makes a true statement from the conditio $ab = c$ is the quotient of replacements for c and a Of course, for the condition $5x = 8$, when $U = C$ the solution set is the empty set because $8 \div 5$ i not a counting number. In unit 7, the student will learn to use a more sophisticated way of find ing solutions of conditions like those we have jus discussed.

There is another way to solve $16/8 \sim 4/x$ Since the first component of $4/x$ can be obtaine from the first component of 16/8 by dividing 1 by 4, the second component of $4/x$ can be ob tained from the second component of 16/8 b dividing 8 by 4. This method is not always sim ple, however. In the condition $25/10 \sim 15/x$ there is no counting number by which 25 can b divided to obtain 15. The 25 can be divided by to obtain 15, but when a counting number canno be used, it is normally easier to use the definitio of equivalent ordered pairs to find the solution o the condition.

Observe that, for the problem in D1 on pag 176, either $16/8 \sim 4/x$ or $8/16 \sim x/4$ could b used as a condition for the problem. These tw conditions are equivalent because the replace ments for x that satisfy the first condition als satisfy the second condition. Since no sequence o events is described in the problem, either condi tion reflects the mathematical structure of th problem.

Instruction notes

■ Begin by asking: "Is the condition expressed i D2 a condition for equality?" [No] "For inequal ity?" [No] "In this lesson you will study condi tions that involve the idea of equivalence."

Before exercise E on page 177, ask: "What es the symbol $16x$ mean?" [That 16 and the re- acement for x are multiplied] "Why is the sign r equality used instead of the sign for equiva- nce in $16x = 32$?" [Because $16x$ and 32 are the me number after x is replaced by the solution] After exercise I on page 177, ask: "Using the te given in the problem, what condition could u use to find how many dollars 14 of the plants st?" [$16/8 \sim 14/x$ or $8/16 \sim x/14$.] "What is e solution of this condition?" [7] "What condi- n could you use to find how many plants can bought for 38 dollars?" [$16/8 \sim x/38$ or $16 \sim 38/x$.] "What is the solution of this con- tion?" [76]

Provision for individual differences

With a class of abler students you can present is lesson by using chalkboard drawings of D1 nd D2 and posing questions similar to those in e developmental exercises.

Have the abler students work the "On your wn" exercises on page 179 and also those below. = C. Tabulate:

1 $\{x \mid x/1 \sim x/1\}$ [{1, 2, 3, . . .}]
2 $\{x \mid 7/1 \sim 2x/12\}$ [{42}]
3 $\{p \mid 32/15p \sim 16/30\}$ [{4}]
4 $\{s \mid s/7 \sim 7/s\}$ [{7}]
5 $\{y \mid 3y/50 \sim 27/6\}$ [{75}]

Have your slow learners use markers to help nem find the solutions of $16x = 32$ and $9y = 27$. hey should see that they divide 32 by 16 and ivide 27 by 9. Emphasize that they must with- raw from the total group enough markers for one round," not one marker at a time. Thus, in nding the solution of $16x = 32$, they first with- raw 16 markers and place one marker in each of 6 groups. Then they withdraw another "round" f 16 markers and again place one marker in ach of the 16 groups.

When you assign the "On your own" exercises, uggest to your students that they arrange their work in a manner similar to that shown in D4, page 177, and D5, page 178. Slower students especially will find it helpful if they are taught to organize their work in a systematic manner.

4 / 44 Exploring problems / Pages 180-184

Problems involving conditions for equivalence

Objectives

■ The students learn to use what they know about rate pairs, equivalent rate pairs, and solutions of conditions for equivalence to find answers to rate and comparison problems.

Content overview

■ The conditions for the problems in this lesson involve equivalence. To solve the problem in D1, we can use a compound condition in two variables, such as $6/8 \sim x/y \wedge x = 24$, or we can use a simple condition in one variable, such as $6/8 \sim 24/y$. A compound condition in two variables requires a universe of ordered pairs. A simple condition in one variable, however, requires a universe of single objects. The solution set of the first condition expressed above is a set of rate pairs, while the solution set of the second condition is a set of single elements. The answer "Tim will have walked 32 blocks" can be obtained from either solution set. The complete answer given in the text reminds students of the full problem situation.

Conditions for equivalence arise from comparison situations as well as rate situations. The components of each ordered pair that represents a comparison situation refer to the same kind of object. Two examples of comparison situations follow: 5 correct answers out of 10 answers and

2 boys who wear glasses for every 7 boys. Each component of 5/10 is a number of answers. Each component of 2/7 is a number of boys.

The components of an ordered pair that represents a rate refer to different kinds of objects. Two examples of rate situations are: 85 miles in 2 hours and $12 for 2 sweaters. The first component of 85/2 is a number of miles, and the second component is a number of hours. The first component of 12/2 is a number of dollars, and the second component is a number of sweaters.

The distinction between a rate and a comparison situation is based on the physical aspects. The mathematical way of handling both situations is the same. While we suggest that you consider making this distinction, it certainly is not crucial, and we hope that you will not test your students on their ability to make the distinction.

After a condition for equivalence is found for a problem, the definition of equivalent ordered pairs is used to obtain a condition for equality. Ideas that we developed earlier are used to find the solutions of the condition for equality.

There are eight more lessons in this book that deal with problems that involve conditions for equivalence.

Instruction notes

■ The following questions suggest ways in which you can emphasize some of the important ideas presented in this lesson.

After exercise A on page 180, ask: "Does each component of 6/8 refer to the same thing?"[Yes] "Turn to D1 and D2 on page 176. Does each component of 16/8 refer to the same thing?" [No] "For a comparison problem, each component of a rate pair refers to the same kind of object. For a rate problem, the components of a rate pair refer to different kinds of objects. Rate pairs are used for both rate and comparison problems."

Before exercise F on page 180, ask: "Does 3/4 represent the same comparison as 6/8?" [Yes] "For the problem in D1, to what does the fir component of 3/4 refer?" [A number of bloc Jerry walks] "To what does the second comp nent of 3/4 refer?" [A number of blocks Ti walks]

After exercise O on page 181, say: "Could v use 8/6 instead of 6/8 in the condition for th problem in D1?" [Yes] "What would the con pound condition be then?" [$8/6 \sim y/x \wedge x = 2$ "How is the condition for this problem differe from the conditions you used in the last lesson? [It is a compound condition in two variables i stead of a simple condition in one variable "Now you will see how you can use either a con pound condition in two variables or a simple co dition in one variable in working rate or compar son problems like the ones in this lesson."

After the problem in D3 is read, ask: "Is th problem about a rate or a comparison?" [Rat the rate at which cars can be washed] Then, aft exercise B on page 181, ask: "Why do we us $w/15$ rather than $15/w$?" [Since the first compo nent of 20/60 refers to a number of cars, the fir component of any other rate pair equivalent t 20/60 also refers to a number of cars.]

Provision for individual differences

■ For a class of abler students, the exercises re lated to the problems in D4, D5, and D6 could b assigned for individual work. Then, for eac problem, you could have a student explain hov he obtained the condition for the problem, hov he found the solutions of the condition, and hov he answered the question in the problem.

You may find that your abler students ca complete the "Applying mathematics" problem on pages 184 and 185 and the "On your own" ex ercises on pages 183 and 184 in one assignment.

Do not expect the slow learners to complet all the "On your own" problems in one assign ment. You might ask them to work the odd numbered problems on the first day and the even

umbered problems on the second day. Discuss me of these problems with the class when the udents start doing the assignment, and suggest students that they develop a systematic method solving these problems. You might set up a attern for them to follow, one that you prefer nd that sets up a step-by-step approach that will elp students become more proficient in problem olving. Such an approach will help students to rganize their work so that they can better follow lass discussion on the assignment.

/ Applying mathematics / Pages 184-185

In these "Applying mathematics" problems, the ollowing terms are introduced: *service charge*, *nterest*, *discount*, *commission*, and *take-home pay*. hese terms are introduced here, rather than in a esson, because they are associated with social nd business applications of mathematical con-epts. They are not mathematical terms.

In this set of "Applying mathematics" prob-ems, the students learn that mathematical con-epts that can be applied to problems about nterest can also be applied to problems about liscount, problems about commission, and prob-ems about take-home pay. We use this approach o avoid the implication that these terms are nathematical terms, an implication that is some-imes made in traditional texts.

Before assigning this set of problems, briefly liscuss with your class the new terms that are ntroduced, but remember that it is not necessary for students to memorize them. Then follow your usual teaching procedure. See pages 83 and 84 of this *Teaching Guide* for further suggestions for using the problems in the "Applying mathe-matics" sections.

4 / 45 Exploring ideas / Pages 186-189

Per cent

Objectives

■ The students learn that a per cent is a rate pair that represents a comparison and has a second component of 100.

Content overview

■ A per cent is a rate pair, and, hence, a per cent is a mathematical idea rather than a symbol. This introduction to per cent provides the groundwork needed to solve per cent problems.

The symbol % is the *per cent sign*. It is used with a numeral to write the symbol for a rate pair that is a per cent. Thus, the symbol 88% is a ratio that expresses a particular rate pair. The rate pair represents a comparison; the first component of the rate pair is 88, and the second component is 100.

In this lesson, we do not specifically teach how rate pairs in general can be ordered, but we do develop the idea of ordering per cents. Because every per cent has a second component of 100, only the first components need be compared. Since $55 < 60$, $55\% < 60\%$ (and $55/100 < 60/100$). Thus, it can easily be seen that $11/20 < 3/5$, since $11/20 \sim 55/100$, $3/5 \sim 60/100$, and $55 < 60$.

We prefer to think of a per cent as a rate pair rather than as a fraction. The following example illustrates the confusion that can be encountered if a per cent is thought of as a fraction. Suppose that, in the first game of a double-header, a ball-player made two hits in five official times at bat in the first game and made three hits in five official times at bat in the second game. For the two games, the comparisons of the number of hits to

the number of official times at bat are 2 to 5 and 3 to 5. 2 to 5 is equivalent to 40 to 100, or 40%, and 3 to 5 is equivalent to 60 to 100, or 60%. His performance for both games together is 5 hits to 10 times at bat, and this is equivalent to 50 to 100, or 50%. Notice, however, that if we consider the fractions $\frac{2}{5}$ and $\frac{3}{5}$, the sum is $\frac{5}{5}$, and if we use the equivalent fractions $\frac{40}{100}$ and $\frac{60}{100}$, the sum is $\frac{100}{100}$. Obviously, this does not state the ballplayer's performance, and we can see that in this situation we would be led astray by confusing fractions with per cents.

We have previously discussed how a rate and a comparison differ. Notice that a rate of 75 cents for 100 pieces of candy can be represented by the rate pair 75/100. This rate pair, however, does not represent a comparison and therefore it is not a per cent.

Suppose that 22/25 and 88/100 each represent the same comparison of money contributed toward a goal. The rate pair 22/25 is not a per cent, since its second component is not 100. 88/100 is a per cent, since this rate pair has a second component of 100 and represents a comparison.

Instruction notes

■ Before beginning the developmental exercises, ask: "So far, what have you learned to represent with rate pairs?" [Rates and comparisons] "Look at D1 on page 186. Does the display picture a rate or a comparison situation?" [Comparison] "In this lesson you will learn to use a special kind of rate pair to represent comparisons."

After exercise F on page 186, say: "Instead of a goal of 25 dollars, we are thinking of a goal of 100 dollars. Instead of 1 group of 25 dollars, we are now thinking of how many groups of 25 dollars?" [Four groups] "Instead of 1 group of 22 dollars, we should now think of how many groups of 22 dollars?" [Four groups] "How many dollars is 4 groups of 22 dollars?" [$88] "Does 88/100 express the same comparison as 22/25?" [Yes]

After exercise K on page 187, have the students use 200 rather than 100 as the second components of rate pairs that represent the comparison of the money raised by each room with the goal for that room. Say: "What condition should you use to find each rate pair?" [$22/25 \sim n/200$; $20/20 \sim n/200$; and so on] "Can you use these rate pairs to compare the performances of each room?" [Yes] "Why not use 65 instead of 100 for the second components?" [Because there is no solution for $22/20 \sim n/65$ when U = C] "Could you use 50?" [Yes] "For comparisons of this kind, people have agreed to use second components of 100."

You could have your students make up a symbol to indicate a rate pair whose second component is 50. For example, 44/50 could be expressed as 44&.

After exercise N on page 187, ask: "How do you know that each rate pair named in exercise N represents a comparison?" [The per cent sign is used only in a ratio that expresses a rate pair that represents a comparison.]

Exercises A through E on page 188 could be assigned for individual work.

Provision for individual differences

■ You should be able to go through this lesson very quickly with abler students. You might discuss the ideas without following the lesson on an exercise-by-exercise basis. Most of the "On your own" exercises can also be covered orally in class. For a written assignment, have your abler students consider problems similar to the following:

1 Room 204 at Jones School wants to collect $80 for a charity. So far they have collected $20. What per cent of the $80 have they collected? U = C. [$20/80 \sim n/100$. {25} 25%]

2 Mrs. James bought a $40 rug on sale for $32. The sale price was what per cent of the original price? U = C. [$32/40 \sim n/100$. {80} 80%]

3 A grocer advertised coffee that usually sells for 90 cents a pound at 5% off. 5% represents a comparison of what amounts? [The number of cents the price is reduced for each 100 cents of the usual price.]

4 Mrs. Adams bought a coat for 80% of the regular price. The regular price was how many dollars for each $12 she paid? U = C. [80/100 ~ 12/n. {15} $15]

5 One evening Jack planned to read 30 pages in a book. He then read 36 pages in the book. The number of pages he read is what per cent of the number of pages he planned to read? U = C. [36/30 ~ n/100. {120} 120%]

With a class of slow learners, you could stop with exercise Z on page 188 and assign "On your own" exercises 1 through 18 on the first day, and then finish the lesson on the next day. If you do this, you may want to provide exercises similar to the "On your own" exercises to ensure that students understand the concepts developed in the lesson. You might ask certain students to give examples of comparison situations that suggest rate pairs. Then have the other members of the class give an equivalent rate pair whose second component is 100 and write or state the answer using the symbol % or the words "per cent."

---

4 / Special challenge / Page 189

Objectives

■ The students develop new techniques of analyzing problems by visualizing different spatial concepts and by making sketches that picture intersections of geometric figures.

Content overview

■ Since these exercises concern space, space should be considered rather than a plane. In exercise A, for example, both endpoints of the segment may be coplanar with the triangle, exactly one endpoint may be coplanar with the triangle, or neither endpoint may be coplanar with the triangle.

For possible solutions, refer to the "Responses to Special Challenges" in this *Teaching Guide*. A sketch may be made to represent the set described in each exercise except C and F.

Exercise C describes an impossible situation. The set that is the intersection of a line and a circle cannot contain three points. The set that is the intersection of a line and a circle can contain no members, exactly one member, or exactly two members. Remember that the center of a circle is not contained in the circle.

Exercise F also describes an impossible situation, because if two planes meet, their intersection is a line, which contains infinitely many members. The intersection of two planes is either the empty set or a line.

---

4 / 46 Exploring problems / Pages 190-193

Problems involving per cents

Objectives

■ The students learn to solve comparison problems that involve per cents.

Content overview

■ A condition for a per cent problem involves rate pairs and the idea of equivalence. The "three cases" that arise in per cent problems can all be handled with one general condition, $a/b \sim c/100$, but we do not use the sentence $a/b \sim c/100$ as a "formula" in *STM 1*. For the problem in D1, the replacement for a is 540; the replacement for b is 720; and the replacement for c is to be found. The

solution of such a condition is found by first using the definition of equivalent ordered pairs to obtain a condition for equality. Thus, the condition $54000 = 720x$ is obtained from $540/720 \sim x/100$. The solution set of $54000 = 720x$ is easily determined.

Our interpretation of the "three cases" of per cent is illustrated below. Notice that the same basic method is used for all three cases. Any per cent problem can easily be solved by using a condition of the form $a/b \sim c/100$. U = C.

x is 20% of 50.	10 is x% of 50.	10 is 20% of x.
$x/50 \sim 20/100$.	$10/50 \sim x/100$.	$10/x \sim 20/100$.
$100x = 1000$.	$50x = 1000$.	$20x = 1000$.
{10}	{20}	{50}

Instruction notes

■ Because the students have learned to solve rate problems, they will have little or no difficulty with problems about per cents. Caution the students, however, to keep in mind that a per cent is a rate pair whose second component is 100 and that a per cent represents a comparison.

The following suggestion may be helpful in presenting the lesson. Before beginning the developmental exercises, say: "Suppose that you were asked to find the per cent of students in this class who are absent today. What numbers would you compare?" [The number of students in the class who are absent with the number of students who are enrolled in the class] "The ideas that you have learned about per cent are important to you only because you can use them to solve per cent problems. Now you will learn how to use these ideas to solve problems about per cent."

After exercise H on page 190, ask: "What comparison is involved in the problem in D2?" [The number of questions answered correctly is compared with the total number of questions.] Consider having your students solve the problem in D3 by themselves. Then have a student explain how he obtained the condition for the problem and the answer to the problem.

Before exercise A on page 191, write on the chalkboard "3/5 ~ 60/100." Ask: "3 to 5 is the same comparison as what per cent?" [60% Write on the board: "3 is 60% of 5." Say: "Do the two sentences express the same idea?" [Yes Then write: "3 is 30% of 10." Ask: "What is another sentence that expresses the same idea?" [3/10 ~ 30/100.]

After exercise F on page 191, ask: "Does $10/6 \sim 100/x$ have the same solution set as $6/10 \sim x/100$?" [Yes] "Is $10/6 \sim 100/x$ a condition that involves a per cent?" [No. A per cent has a second component of 100.]

Provision for individual differences

■ For a class of abler students, the following developmental exercises might be omitted:

N through U, page 191

U through Y, page 192

Have these students work the problems in D3 and D9 without using the developmental exercises and then explain their work in class discussion. They may be able to complete the "On your own" problems and part of the "Applying mathematics" problems on pages 194 and 195 as one assignment.

For slow learners, you might assign some of the odd-numbered problems for one day's assignment and part of the even-numbered problems for another day. Help them organize their work. You might suggest a particular format that you prefer them to use in a problem-solving situation.

---

4 / Applying mathematics / Pages 194-195

■ In these "Applying mathematics" problems, the following business and social terms are intro-

luced: *rate of commission*, *rate of discount*, *princi-al* and *rate of interest*. Remember that these are ot mathematical terms, but terms that are asso-iated with applications of mathematical con-epts.

See page 129 of this *Teaching Guide* for a dis-ussion of social and business terms. See also ages 83 and 84 of this *Teaching Guide* for sug-gestions for using "Applying mathematics" prob-ems.

Problems involving compound conditions about rate pairs

Objectives

■ The students learn to solve problems about ates and comparisons that involve compound conditions.

Content overview

■ For each of these problems, one of the simple conditions of the compound condition is a condition for equivalence. The other simple condition is either a condition for inequality or a condition for equality. Note that the condition for a problem may involve three simple conditions in three variables, such as the condition for the problem in D4.

Different conditions can often be used for the same problem, but certain conditions are preferable to others. The following sentences express three conditions for the problem in D3:

$$273 - 260 = n \wedge n/260 \sim m/100.$$
$$273 - 260 = n \wedge m/100 \sim n/260.$$
$$n/260 \sim m/100 \wedge 273 - 260 = n.$$

An example of a condition that has the same solution but is not acceptable for the problem is:

$$273 - 260 = n \wedge 260/n \sim 100/m.$$

This last condition does not indicate that a per cent is involved.

If a compound condition has a simple condition that involves just one variable, it is usually more convenient to first find the solutions of this simple condition and then use these solutions as replacements for the same variable in the other simple condition.

The problem in D4 is an example of a "multiple-step" problem. Actually, any problem that involves two or more requirements is a "multiple-step" problem. Now such problems can be studied as a part of the application of more mature mathematical ideas and symbolism. The problem in D4 can be solved without using a mathematical condition. However, we use mathematical methods to solve simple problems such as this one to develop for the students the ideas that they will need to solve more difficult problems later on.

Instruction notes

■ After the problem in D1 is read, ask: "What two requirements in this problem must be satisfied by the answers to the problem?" [The number of records Jane has is exactly four times as great as the number of records Sally has, and Jane has fewer than 25 records.] "Why should we use a compound condition for this problem?" [Because the numbers we are to find must satisfy two requirements]

Before exercise F on page 196, ask: "Could we use $25 > x$ rather than $x < 25$?" [The solutions are the same, but since the problem involves "less than," $x < 25$ better interprets the situation described in the problem.] "What other conditions could we use for the problem in D1?" [Two examples are: $1/4 \sim y/x \wedge x < 25$ and $x < 25 \wedge x/y \sim 4/1$.]

If you did not assign problem 3 of the "Applying mathematics" on page 184 and problem 10 on

page 194, turn to these problems and use them for a brief review of the topic of interest before beginning the developmental exercises after exercise K on page 196.

After exercise R on page 197, ask: "Why is it a good idea first to find the solution of the simple condition $273 - 260 = n$?" [The solution can be found mentally. $m/100 \sim n/260$ has many solutions.] "Think of a compound condition in which one simple condition is in two variables and the other simple condition is in one variable. The solutions of which simple condition are usually easier to find?" [The condition in one variable]

"Look again at exercise F on page 196. For which simple condition did we first find solutions?" [The simple condition in two variables] "Could we have found first the solutions of the simple condition in one variable?" [Yes] "What are some solutions of $x < 25$?" [(24, 1), (24, 2), (24, 3), (23, 1), (23, 2), (22, 1), and (22, 2) are only a few of the solutions.] "Could we use some of these solutions to find the solutions of the compound condition?" [Yes. We could use each of the numbers from 1 through 24 as replacements for x in $4/1 \sim x/y$ and then find a replacement for y, if any, that makes a true statement.]

Encourage students to use care in identifying the variables before they attempt to write a sentence that expresses a condition for a problem. Also, since the problems have been carefully planned to help students gain experience before dealing with more difficult problems, they should always write a sentence that expresses a condition for a problem before they find solutions.

Provision for individual differences

■ A class of abler students might complete lessons 47 and 48 in one day's work. To do this, you should plan a class discussion patterned after the developmental exercises.

You might assign the following exercises to the abler students:

1 Name three replacements for (m, n) that satisfy $m/100 \sim n/260$. $U = C \times C$. [Examples are: (5, 13), (10, 26), (15, 39).]

2 For the problem in D4, we used a as a variable for the number of cents Jim spent for pencils and n as a variable for the total number of cents Jim spent. To what does $n - a$ refer? [The number of cents Jim spent for erasers] To what does $n - (n - a)$ refer? [The number of cents Jim spent for pencils]

For a class of slow learners, you might consider omitting the exercises that deal with the problem in D4 (the last block in the lesson) and thus reduce the time spent on the lesson. If you decide to omit these exercises, do not assign "On your own" problem 13.

The slow learners, if properly motivated, however, may gain a great deal of satisfaction from working with this problem. Suggest that they explain to their parents how to find a solution for the problem using a sentence similar to the one given in exercise J on page 197. A student who finds that a parent is impressed with our problem-solving approach will be strongly motivated to learn more about this method. Also, the student will find that it will be necessary for him to have his work well organized in order to explain to his parents how to find the solution for the problem. This should strengthen your request that students follow a pattern or model in organizing their work.

---

4 / 48 Exploring problems / Pages 199-201

Abstract problems involving rate pairs

Objectives

■ The students learn to solve abstract rate problems that have two requirements.

Content overview

Each of the problems in D1, D3, and D5 involves a compound condition in which the solutions of both simple conditions may be found by tabulating the first few solutions of the simple condition for equivalence and then determining which of these solutions also satisfy the other simple condition.

The answer to an abstract problem is given by a tabulation of the solution set of the condition for the problem. The answers are numbers or rate pairs rather than quantities.

Instruction notes

You could begin this lesson by asking the students to refer to D1 on page 195. Have them express this problem as a problem that involves only numbers, rather than numbers of phonograph records. [Certain rate pairs are equivalent to 4/1. The first component of each of these rate pairs is less than 25. What are the rate pairs?] Then have the students develop a condition and find its solutions.

After exercise H on page 199, you could have the students finish the lesson on their own. Then discuss the problems in D3 and D5, the conditions for those problems, and the solutions of the conditions.

You might have students make up other abstract problems and write them on the chalkboard for the class to solve. You might also have a student make up a compound condition in which one simple condition is a condition for equivalence. Then have the students make up problems for this compound condition.

Provision for individual differences

For a class of abler students, you may find that you need to use only the developmental exercises that are related to the problem in D1 (exercises A through H).

For a class of slow learners, this lesson could be omitted. But if you wish to use it for these students, it would be advisable to give detailed attention to the developmental exercises and to give much individual help with the "On your own" problems.

5 / 49 Exploring ideas / Pages 206-211

Tally, code, and grouping numeration systems

Objectives

The students study tally, code, and grouping numeration systems.

Content overview

We believe it is important to make a distinction between a number system and a numeration system.

A *number system* is a set of mathematical ideas, together with certain operations and properties of the operations. The system of natural numbers is developed in unit 6, and the system of rational numbers of arithmetic is developed in unit 7.

A *numeration system* is a method, or a system, for naming numbers. In this unit we present simple numeration systems first. Then we present the fundamentals of the systems of numeration used by the ancient Babylonians, Egyptians, and Romans. With this approach, the students have a background to more fully understand the properties of the decimal system of numeration.

A numeral is defined as "a name of a number," and, hence, any symbol that expresses a particular number is a numeral for that number.

One system of numeration is a *tally system* of numeration. In a *tally numeral*, there is a one-to-one correspondence between the set of strokes and a set associated with the number named by the numeral. Therefore, the number of strokes in

a tally numeral is the same as the number of objects. Thus, to name a number greater than one in a tally system, more than one stroke is used. Since the stroke may be used more than once in a numeral, a tally system has the property of *repetition*. Note that some primitive numeration systems do not utilize a symbol for zero.

A tally numeral, then, is a numeral that is written by using these simple rules and properties of a tally system of numeration. Because a tally numeral expresses the sum of the ones that are represented by strokes, a tally system has the property of *addition*. A tally system does not have the properties of place value or grouping.

In a *code system* there is an arbitrary association between each symbol and a number. You cannot interpret one symbol in terms of another by using properties such as repetition, grouping, base, and so on.

A *grouping system* of numeration has the properties of addition, of repetition, and of base. A grouping system also involves code numerals. A grouping system has a symbol for the number 1 and a symbol for each other power of the base.

In a base-five grouping system, a symbol is used for the number 1, a different symbol for the number 5, still another symbol for 5×5, or 25, and so on. To express the number 39, we would write one of the symbols that represents 5×5, or 25; two of the symbols that represent 5; and four of the symbols that represent 1. The symbols may be written in any order because a grouping system does not have the property of place value.

It is also possible to express the number 39 in a base-five grouping system by a numeral that contains thirty-nine symbols for 1, or by a numeral that contains six symbols for 5 and nine symbols for 1, and so on. When we agree, however, to express a number by using as few symbols as possible, writing the numeral is more concise, and reading the numeral is easier.

Instruction notes

■ You will find that most students enjoy studyin numeration systems. You could introduce th work of this unit and teach this lesson in the fo lowing way. Write on the chalkboard "4, fou IV, ////." Say: "Does each of these symbol name the same number?" [Yes] "In how man ways can the number four be named?" [Man ways] "In this unit you are going to study differ ent systems for naming numbers. You will fin new ways to name numbers."

You could present the tally system and it properties without using the book. Invent a imaginary situation, such as a trip in a space shi in which you might use tally marks to keep a record of the number of earth days. You woul need a 24-hour clock, since there would be n night and day to help you.

After presenting the ideas in the first half o column 1 on page 207, write on the board the tw numerals "//" and "11." Ask: "Does this tall numeral express one plus one, or two?" [Yes "What does the decimal numeral express?" [Te plus one, or eleven] "Does the decimal syste have at least one property that a tally system doe not have?" [Yes, place value]

Before discussing exercise K on page 208, ask "If we change the order of the symbols in a grouping numeral, does the numeral we obtai express the same number as the original nu meral?" [Yes] "Think about whether or not thi is true in the decimal system. Does the numeral 2 express the same number as the numeral 32?" [No]

Provision for individual differences

■ Have the abler students make a chart like th one on the next page.

Tell the students that they are to complete thi chart for grouping systems in the bases listed a the top of the columns. Suggest that they invent symbols for each power of the base.

lumber	Base two	Base three	Base eight	Base ten	Base twelve
ne	/				
vo	★				
ıree	★ /				
our	◈				
ve	◈ /				
x	◈ ★				
even	◈ ★ /				
ight	◐				
ine					
en					
leven					
welve					
hirteen					
ourteen					

For a class of slow learners, on the first day e the exercises that begin on page 206 and continue through the first paragraph that follows ercise N on page 209. As a written assignment, ave these students use a stroke for 1 and a star r 4 to write a grouping numeral for the number pressed by each tally numeral in exercises I rough R on page 207 (column 1). [I, /; J, / /; , / / /; L, ★; M, ★ /; N, ★ / /; O, ★ / / /; , ★ ★; Q, ★ ★ /; and R, ★ ★ / /] Also, have e students follow the same procedure for "On our own" exercises 1 through 10 on page 210, sing a stroke for 1, a star for 4, and a diamond r 16. Caution your students to follow your directions rather than those given for the exercises. , ★ / /; 2, / /; 3, ★ /; 4, ★ ★; 5, ★ ★ ★ / /; 6, ★ ★ ★; 7, ★ ★ / / /; 8, ◈; 9, ★ / / /; 0, ◈ ◈ ★ ★ ★ /]

On the second day, carefully check the written ssignment, finish the lesson, and assign the rest f the "On your own" exercises.

The student coming to the seventh grade has n extensive background in the decimal system of numeration, and he will be familiar with many of the ideas developed in unit 5, such as tally and grouping numeration systems, place value, and the Roman numeration system. Omitting some of the lessons in this unit will not interfere with the student's understanding of the important mathematical ideas presented in following units.

If your class is not able to maintain the time schedule you have planned, you can omit lessons 49, 50, 51, 52, 54, and 55. Lesson 53 should *not* be omitted because the ideas and the notation developed in this lesson are utilized in succeeding lessons. Lesson 56 may be omitted only if students have a good understanding of decimal numerals for numbers greater than one thousand.

5 / 50 Exploring ideas / Pages 211-214

Numeration systems that have place value

Objectives

■ The students learn about numeration systems that have the property of place value as well as the properties of addition, repetition, and base.

Content overview

■ A simple place-value numeration system may be devised that uses a stroke as a symbol for one, the properties of addition and repetition that are used in a tally system, and the idea of base that is used in a grouping system. Unlike a grouping system that employs a unique symbol for each power of the base, however, the powers of the bases of the systems in this lesson are associated with the different positions of the strokes.

The choice of vertical red rules to separate the strokes in each position in a numeral was arbitrarily made. The strokes in each position could

be separated by asterisks, hyphens, or other suitable separation marks.

The decimal numeration system, developed in the next lesson, has the properties of the systems used in this lesson. With the decimal system, however, we use one code symbol (instead of a set of strokes) in a position to indicate the number of times a power of the base is expressed by the numeral.

Precisely speaking, a numeral does not have properties. A system of numeration has properties such as addition, repetition, and so on. However, a numeral may indicate some of the properties of the system of numeration of which it is a part. Numeral A in D1 is obviously written in a system that has the property of repetition. The paragraph that follows exercise C on page 211 gives meaning to the questions asked in exercises C and D. This explanation makes it unnecessary to ask the longer and more formidable-sounding question "How do you know that the numeration system in which numeral A is written has the property of repetition?"

Instruction notes

■ Before beginning the exercises, ask: "In the last lesson, how many symbols did you use in the grouping systems?" [4 symbols: stroke, star, diamond, arrowhead] "If you named greater and greater numbers, would you need more symbols?" [Yes] "Suppose that you wanted to express one billion in a base-four grouping system. How many different symbols do you think you would need?" [Let a few students guess. They would need 15 symbols because 15 places would be required to express one billion using a base-four numeral, if we use as few symbols as possible.] "In this lesson you will learn about another system of numeration that uses just one symbol. This system has the properties of addition, repetition, base, and one more property." Then begin the lesson.

While your class works out the answer to exercise B on page 211, write on the chalkboard "27 + 18 + 6 + 2 = 53." Ask: "In obtaining the numeral 53 from numeral A, did we convert a grouping numeral to a decimal numeral?" [Yes]

Before discussing exercise K on page 212, have your students express the number 71 in a base-three grouping system by using strokes, stars, diamonds, and arrowheads. [◐ ◐ ◈ ★ ★ / /] Then have them express the same number in a base-three place-value system by using strokes. [/ / | / | / / | / /]

After exercise T on page 214, say: "When you add numbers, using decimals for your computation, do you regroup numbers?" [Answers may vary.] Write on the board the decimal numerals shown at the right. "Suppose that we want to find the sum of the numbers expressed by these decimal numerals. How many ones are there?" [13] "Do I write the numeral 13?" [No] "Do I have to think of regrouping ten of the ones into one group of ten, which I will then add to the other tens?" [Yes]

2
3

Provision for individual differences

■ All of the developmental exercises in this lesson should be considered by all of the students in the class.

Abler students will be interested in learning how an abacus is used. Some of the students could make an abacus. If a student is willing to make one, suggest that he make it in base five, base six, or base seven. An abacus utilizes the idea of place value to express numbers.*

* Four of the many books that contain informative articles about the abacus are:
Donovan A. Johnson and William H. Glenn, *Understanding Numeration Systems*, in the series *Exploring Mathematics On Your Own* (St. Louis: Webster Publishing Company, 1960).
Foster E. Grossnickle and Leo J. Brueckner, *Discovering Meanings in Elementary School Mathematics* (Toronto: Holt, Rinehart and Winston, 1963).
Robert L. Swain, *Understanding Arithmetic* (Toronto: Holt, Rinehart and Winston, 1960).
Collier's Encyclopedia (New York: Crowell-Collier, 1960), I, 4.

You also could have your abler students find e difference and the product of two numbers pressed by the kind of numerals they studied in is lesson. For example: "Subtract / /|/ / /|/ / om / / /|/ /|/. The numerals are written in a se-five system." Then let them check their an- ers by converting to decimal numerals and per- rming the computation again.

To provide further practice for slow learners, k them to think of the numerals in exercises 1 rough 10 of the "On your own" exercises as se-ten numerals. Have them write decimal nu- erals (using digits) that express the same num- rs. For example, for exercise 1 the students ould write "$1(10 \times 10) + 4(10) + 3(1)$, or 143."

/ 51 Exploring ideas / Pages 215-218

lace value and digits n numeration systems

Objectives

The students learn that bases other than ten can e used to develop numeration systems that have ne same properties as the decimal system.

Content overview

The numeration systems developed in this les- on have the properties of the systems presented n the last lesson. In the systems in this lesson, owever, a code symbol (instead of a set of trokes) is used in each position in a numeral.

Modern systems have a number of advantages ver the place-value systems used in lesson 50. ome of these advantages are: Numbers can be xpressed very concisely; no special symbol such s a vertical bar is needed to separate two adjoin- ng positions; it is easy to determine the number numeral expresses; computation is simple; and here is a digit for zero.

Historically, the digit 0 was first used as a placeholder for a position in a numeral. From this point of view, the decimal numeral 204 would mean $2(10 \times 10) + 4(1)$. In *STM 1*, we take the view that the digit 0 does, in fact, express a number. Our present expanded form of the decimal numeral 204 is $2(10 \times 10) + 0(10) + 4(1)$.

A digit is a code symbol that expresses a unique number. For example, the digit "2" refers to the number two. The number of different digits needed in a numeration system like those presented in this lesson is the same number as the base of the system. Thus, our decimal system of numeration has ten digits, 0, 1, . . ., 9. In a binary (base-two) system, only two symbols are needed. It is convenient to use the symbols 0 and 1. In a duodecimal (base-twelve) system, twelve symbols are needed. In *STM 1*, we use the symbols 0, 1, . . ., 9, and $\mathcal{X}$ ("dek") as a code symbol for the number ten and $\mathcal{E}$ ("el") for eleven.

The base-four system is presented before the decimal system in this lesson so that the students can view the decimal system from a new perspective.

The numeral 33_{four} should not be read as "thirty-three, base four" because this name suggests the number thirty-three. Actually, the number fifteen (in base 10) is expressed by the symbol 33_{four}. In *STM 1*, the numeral 33_{four} is read "three three, base four."

Instruction notes

■ We suggest that you follow the developmental exercises in presenting this lesson.

Before exercise P on page 216, say: "Think about the words 'One thousand thirty-three.' Are these the words you would use to read a decimal numeral?" [Yes] "Is the numeral in D3 a decimal numeral?" [No] "We should use words for decimal numerals only when we are reading or talking about decimal numerals. When we are talking

about numerals written in other systems, we should just say the name of each digit, in order, and then identify the base of the system."

Following the first paragraph after exercise M on page 217, say: "Look again at numerals A and B in D5. Are both of these numerals expressed in a base-ten system of numeration with place value?" [Yes] "Of these two numerals, only B is considered a decimal numeral. The decimal system of numeration has ten code symbols. Even though numeral A is written in a base-ten system with place value, it is not considered a decimal numeral, because it has only tally symbols."

After exercise M on page 218, say: "We have used two symbols for the base-twelve system that most of you were not familiar with. Do you think that man invented the code symbols we use in the decimal system?" [Yes] "One thousand years ago, the symbols that were used for 2, 3, 4, 5, and 6 were different from the symbols we use for those numbers today."

Provision for individual differences

■ With a class of slow learners, you may find it helpful to emphasize one numeration system, for example, base four, and provide students with numerals in this base that are to be "translated" into base-ten numerals. Have the students make a chart similar to the chart in D1 on page 215. Such charts emphasize the meaning of place value.

When you feel that students have an understanding of the general properties of a positional, place-value system, have them work the "On your own" exercises. Slow learners will find it helpful to continue to make and use charts similar to those in D1, D5, and D7 when doing these exercises.

Abler students could consider an additive alphabetic numeration system similar to that used by the early Hebrews and Greeks. Each letter of the alphabet expresses a number. To find the number expressed by a combination of lette the numbers expressed by the letters are adde Suppose that A expresses 1, B expresses 2, C e presses 3, D expresses 4, and so on. Using th system, the word BELL expresses 2 + 5 + 12 + 1 or 31. Have your abler students answer the fo lowing questions:

1 Using this system, what number is express by your name? [Answer varies with name.] Ho would you express the number of years that your age? [13 is expressed by M.]

2 What number does each of the following wor express in this numeration system?
apple [1+16+16+12+5, or 50]
house [8+15+21+19+5, or 68]
zoo [26+15+15, or 56]
potato [16+15+20+1+20+15, or 87]
mathematics
[13+1+20+8+5+13+1+20+9+3+19, or 112]

3 If we use this system, does "great < less" e press a true statement? [Yes. 51 < 55.]

4 Does the system you used for exercises 1, and 3 have the property of addition? [Yes] C repetition? [Yes] Of place value? [No]

---

5 / Special challenge / Page 218

Objectives

■ The students use what they have learned abou solving problems involving compound condition that contain two variables. They solve two diffi cult problems.

Content overview

■ To solve problem A, we must first find th amount Mr. Ames paid for the two cars. The con dition $80/100 \sim 528/x$ can be used to find th amount he paid for one car, and $120/100 \sim 528/$ can be used to find the amount he paid for th

ıer car. $x + y = n$ can be used to find the ıount he paid for both cars. By using the defi- ion of equivalent ordered pairs and division d addition, we find that Mr. Ames paid $440 one car, $660 for the other car, and $1100 for th cars. We can use $528/1 \sim k/2$ to find the ıount for which he sold both cars. Then we can e $1100 - 1056 = m$ to find that Mr. Ames lost 4. Observe that overhead is not considered.

For problem B, we can use l for the weight of e lightest rock; m for the weight of the second- htest rock; and h for the weight of the heaviest ck. The compound condition for the problem $l + m = 14 \wedge l + h = 15 \wedge m + h = 19$. By ex- ining $l + m = 14 \wedge l + h = 15$, we can see that e heaviest rock weighed one pound more than e second-lightest rock. Therefore, $h = m + 1$, d we can replace h by $m + 1$ in $m + h = 19$ to tain $m + (m + 1) = 19$. Then we know that $n + 1 = 19$, $2m = 18$, and $m = 9$. We can replace by 9 in $l + m = 14$ to obtain $l + 9 = 14$ and = 5. We can replace m by 9 in $m + h = 19$ to tain $9 + h = 19$ and $h = 10$. Then we know that e lightest rock weighed 5 pounds, the second- htest rock weighed 9 pounds, and the heaviest ck weighed 10 pounds.

52 Exploring ideas / Pages 219-223

gyptian, Roman, and Babylonian umeration systems

Objectives

The students compare ancient numeration sys- ms with the decimal system of numeration.

Content overview

After many centuries of experience with other stems of numeration, man evolved the decimal stem of numeration.

The three systems discussed in this lesson were rather widely used in ancient times. The Egyptian system dates back to about 3500 B.C., the Babylonian system to about 2000 B.C., and the Roman system to about 500 B.C. The Roman system was widely used long after the time of Christ and, of course, is used today for special purposes.

The ancient Egyptian numeration system was a base-ten grouping system. Because it was a grouping system, it had the properties of addition, repetition, and base.

The Egyptian symbol for 100 is a picture of either a scroll or a coil of rope. The symbol for 1000 is a stylized picture of a lotus flower. The symbol for 10,000 is a picture of either a bent reed or a bent finger. While some Egyptians used the symbol for 100,000 that is pictured in D1 on page 219, others used a symbol that looks something like a picture of a tadpole, which they called a "burbot." The symbol shown for 1,000,000 is referred to as an astonished man.

The ancient Romans used a modified grouping system. They had symbols for the first four natural-number powers of ten (I, X, C, and M), along with symbols for the product of five and each of the first three natural-number powers of ten (V, L, D). Thus, the Roman system is generally regarded as a "mixed-base" grouping system.

The Romans used the properties of addition and repetition in their system. At times, some Romans also used a subtraction property in expressing certain numbers. Using this property, a Roman wrote the numerals IV, XL, and CD to express four, forty, and four hundred, and he wrote the numerals IX, XC, and CM to express nine, ninety, and nine hundred. The use of the property of subtraction was generally limited to these special cases to avoid ambiguity. There is evidence that four was also expressed by the nu-

meral IIII, ninety by the numeral LXXXX, and so on.

To express very great numbers, Romans reputedly used a device not presented in this lesson, the use of a bar over a numeral to express a number one thousand times as great as the number normally expressed by the numeral. Thus, $\overline{L}$ could be used to express 50,000, and $\overline{M}$ could be used to express 1,000,000.

The Romans made a practice of performing their computation with an abacus.

The Babylonians, who had a highly developed culture for their time, used a sexagesimal system of numeration—a place-value system with a base of sixty. This system had the properties of addition, repetition, base, and place value. Two code symbols were used, a symbol for 1 and a symbol for 10. The place-value positions in a numeral were indicated by the spacing of each group of symbols.

The Babylonians formed their numerals with a stylus in clay tablets. This is known as cuneiform writing.

Of course, many other systems of numeration were used in ancient times. The Hebrews, and later the Greeks, used the letters of their alphabets as numerals. The Mayans used a system of numeration with a base of twenty and a symbol for zero.

It is not particularly difficult to perform addition and subtraction computations using the ancient numeration systems, but division must be performed as repeated subtraction, and this can be a very tiresome process when only the number in one group is subtracted at a time.

The study of these ancient numeration systems should help your students understand the evolution of the decimal system. Since this is the real purpose of the present lesson, it is not imperative that your students retain the details of what they learn here. The tests in this book contain no items about the Egyptian or the Babylonian s tems and very few items about the Roman s tem. Since this lesson has more historical th practical significance, your students need r memorize the symbols used in the Egyptian or t Babylonian systems.

Instruction notes

■ Say: "In this lesson you will learn about thr numeration systems that were used more th two thousand years ago."

After exercise G on page 220, ask: "Was t Egyptian numeration system a base-ten groupi system?" [Yes] "The Egyptian system had t same properties as the grouping systems y studied in lesson 49."

Before exercise L on page 220, say: "You ha found that you can add using the Egyptian sy tem. You had to regroup the numbers after y combined the symbols.

"Suppose that 5000 years ago an Egypti wanted to know the total number of goats in t flocks when each flock contained 24 goats. Ho do you think the Egyptian would have solved th problem?" Let the students make guesses. [Th Egyptian probably would have used some form repeated addition to obtain the answer to th problem.]

Provision for individual differences

■ A class of abler students may omit the follow ing developmental exercises:

F, K, L, page 221

Suggest to the abler students that they use e cyclopedias or other available reference books t find out more about the systems of numeratio studied in this lesson. Also have them look u other systems, such as the Mayan and the ancie Hebrew and Greek systems. Encourage them t give reports on their work.

Do not insist that students gain a mastery o the systems studied in this lesson. With a class o slow learners, develop the lesson as a simple su

y of ancient systems of numeration, and treat
htly the exercises that involve computation.
nphasize the Roman numeration system.

53 Exploring ideas / Pages 226-229

ictors, powers, and exponential notation

Objectives

The students study the ideas of factor, expo-
nt power, and exponential notation. They also
arn what is meant by writing numerals in ex-
nded form.

Content overview

The ideas and definitions presented in this les-
n are basic to the work that follows in the next
ree lessons. The idea of factor is further devel-
ped in lesson 69; the ideas of factor and power
e used in lesson 70; and the ideas of power and
panded form of a numeral are more fully de-
loped in lesson 86. These ideas are extremely
nportant to students for much of their future
ork in mathematics.

In addition to the definition of *factor* that is
ven in the text, consider the following: "In gen-
al, the replacement for a is a factor of the re-
acement for c if and only if there exists a re-
acement for b such that a true statement is
btained from $ab = c$. For each variable, U = C.
or each true statement obtained from $ab = c$,
ie replacement for b is also a factor of the re-
lacement for c." We feel that this definition is
o formal for most seventh graders.

Using the definition of factor, if c is replaced
y 6, then the solutions of $ab = 6$ are (1, 6), (6, 1),
, 3), and (3, 2). Therefore, the set of factors of 6
{1, 2, 3, 6}.

Notice that in this lesson the term factor is re-
ricted to counting numbers. (Thus, $\frac{1}{2}$ does not have factors as the term is used here.) This use of the word factor is correct, but factor can also refer to numbers like $\frac{1}{2}$, π, sin 30, and so on. $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ indicates that $\frac{1}{2}$ and $\frac{1}{3}$ are factors of $\frac{1}{6}$. This meaning of factor is introduced in *STM 2*, and from then on the context will indicate which definition of factor is to be used.

The numeral 10^3 represents a power of 10. The numeral 1000 expresses the same power of 10. Thus, 1000 is the same number as 10^3. The numeral 10^3 is in exponential form, and the numeral 1000 is in decimal form.

For 10^3, the base is 10. The base of a power is the number that is used as a factor to obtain the power. For 10^3, the exponent is 3. The exponent is the number of times the base is used as a factor.

The zero power of any counting number is the number 1. The zero power of zero is indeterminate, although, for special purposes, mathematicians may *define* the zero power of zero as the number 1.

Exponential notation, which is a concise way of expressing a number, is used in writing the expanded form of a numeral and in scientific notation (see lesson 56). An understanding of exponential notation is useful for future work with logarithms.

The expanded form of a numeral indicates an analysis of the meaning of the numeral. In *STM 1*, we have limited the meaning of expanded form to make it easy to understand and also to prevent ambiguity. In this book an expanded form is always a decimal numeral, and, whenever possible, exponential notation is used. For example, the expanded form of the decimal numeral 543 is the numeral $5(10^2) + 4(10^1) + 3(1)$, and the expanded decimal form of the base-six numeral 543 is the numeral $5(6^2) + 4(6^1) + 3(1)$. To express the base-six expanded form for the numeral 543_{six} without using decimal numerals, we write the numeral

$5(10^2) + 4(10^1) + 3(1)$. In this case, 10^1 is another name for six and 10^2 is another name for thirty-six.

Ordinarily, the common symbols from the decimal system are used in place-value systems of other bases. However, for any place-value system, the symbols used are arbitrary. The familiar symbols are, of course, easier to use, since it is not necessary to develop an understanding of the meaning of the basic symbols 0, 1, 2, and so on. For a base-four system, we could use the symbols √, −, +, and #, but we would have to decide which of the first four natural numbers each of these symbols represents. If we conclude that √ = 0, − = 1, + = 2, and # = 3, then the numerals for the first ten natural numbers are: √, −, +, #, −√, −−, −+, −#, +√, and +−.

Instruction notes

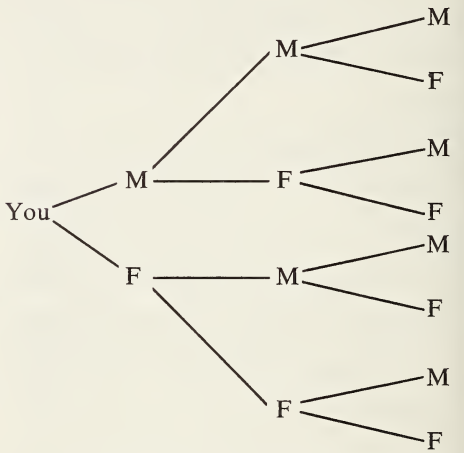
■ After exercise M on page 226 has been discussed with the students, write on the chalkboard the example at the right. Say: "What can you do to make this division example easier?" [Divide both the dividend and the divisor by 100 and indicate this division by crossing out two of the zeros in each numeral, starting at the right of each numeral.] "When you perform this division by 100, are you really dividing by a factor that 15,000 and 500 have in common?" [Yes] "Now you can see that this is the idea of factor that you used in the sixth grade."

$500\overline{)15000}$

Before discussing exercise N on page 227, put the following chart on the board:

Number of generations back	Number of ancestors
1	Parents
2	Grandparents
3	Great-grandparents
4	Great-great-grandparents

Say: "Suppose that you are thinking about th greatest number of ancestors a person could hav Can you write the numeral for the first power o two to express the number of parents a perso has?" [Yes] Write "2, or 2^1" at the right of th numeral 1 on the chart. "How many grandpa ents could a person have?" [4, or 2^2] Indicate th on the chart. If your students have trouble figu ing out the number of great-grandparents, make "tree diagram" that represents the arrangemen of mothers and fathers. See the diagram below:



One generation back | Two generations back | Three generations back

"How many ancestors could a person hav from four generations back?" [2^4] "How many ancestors could a person have from eight generations back?" [2^8] "How many ancestors could person have from n generations back?" [2^n]

After exercise A on page 228, ask: "Is the third power of 2 the same number as the second power of 3?" [No. $2^3 = 8$ and $3^2 = 9$.] "Exercise A gives an example in which 4^2 and 2^4 are the same number. Can you think of another example like the one in exercise A?" [Let the students try to find another example. There is no other similar case.]

Before exercise I on page 228, ask: "In what ıy are numerals B and C in D5 different?" [Exnential notation is used in numeral C, but it is t used in numeral B.] "Remember that expontial notation is used in the expanded form of a meral."

Provision for individual differences

For a class of abler students, the following delopmental exercises can be omitted:

I through M, page 226
G through K, page 227

For a class of slow learners, use two periods r this lesson. On the first day, discuss the lesson rough exercise N on page 227 and assign "On ur own" exercises 1 through 21. On the second ıy, finish the lesson and assign "On your own" ercises 22 through 39. Do not try to get the slow arners to generalize or verbalize too quickly.

Abler students could consider the following ercises in addition to the ones in "On your vn." U = C. Tabulate:

1 $\{x \mid x^7 = 1\}$ [{1}]
2 $\{y \mid 1^y = 85\}$ [{ }]
3 $\{p \mid 2^p = 4^2\}$ [{4}]
4 $\{n \mid 9 \times 2^n = 6^2\}$ [{2}]
5 $\{t \mid 2^t > 2\}$ [{2, 3, 4, . . .}]
6 $\{a \mid 3^a < 3^4\}$ [{1, 2, 3}]

Use a decimal numeral to express each number amed in exercises 7 through 12.

7 $2(7^3) + 4$ [690]
8 $(7^2)^1$ [49]
9 $(2^3)^2$ [64]
10 $(2^2)^3$ [64]
11 $(3^2)^3$ [729]
12 $2^2(3^3)$ [108]

Write "T" for each sentence that expresses a ue statement for each replacement of the varibles. Write "F" for each sentence that expresses false statement. U = C.

13 $a^b > 0$. [T]
14 $a^3 = 3^a$. [F]
15 $2(a^3) = a^3(2)$. [T]
16 $a^3 + a^2 = a^{3+2}$. [F]
17 $a^3(a^2) = a^{3+2}$. [T]
18 $5^a(5^b) = (5^a)^b$. [F]
19 $2a^2(3a) = 6a$. [F]
20 $(ab^2)c = (b^2a)c$. [T]
21 $a^3(a^3) = a^6$. [T]
22 $a + a^2 = a^{1+2}$. [F]

5 / 54 Exploring ideas / Pages 229-232

Converting numerals from one base to another

Objectives

- The students learn to convert numerals from one base to another.

Content overview

- Numerals A, B, and C in D1 on page 229 all express the same number, but numeral C is the expanded form of the numeral. Observe that, when we say "expanded form" of a numeral, we mean "the *base-ten* expanded form" of the numeral.

We feel that at this level students acquire a better understanding of non-decimal numerals if they think of them in terms of what is familiar to them—the decimal system. When students convert non-decimal numerals to decimal numerals and decimal numerals to non-decimal numerals, they think in terms of the familiar decimal system of numeration. Even when performing computation with numerals in non-decimal systems, the students think in terms of decimal names for numbers (if they are not using tables). For example, when given $3_{\text{five}} + 4_{\text{five}} = x$, the students probably think $3 + 4$ is 7 and 7 is $1(5) + 2(1)$. Therefore, the solution of the condition will be named by the symbol 12_{five}. When the students think of seven, they are thinking in terms of the decimal name of the number.

More precisely, "converting a numeral" means using another system of numeration to write a numeral that names the same number. While a numeral can be converted in many ways, the ways presented in this lesson are mathematically sound and are not difficult to understand.

A way to convert the decimal numeral 97 to a base-five numeral (other than the way shown on page 231) is as follows:

(1) Divide 97 by 5. ⟶ 19 remainder 2

(2) Divide 19 by 5. ⟶ 3 remainder 4

(3) Divide 3 by 5. ⟶ 0 remainder 3

Using the remainders obtained, we write the base-five numeral for 97 as 342_{five}.

Instruction notes

■ It is not important that students achieve complete mastery of the ideas presented in this lesson. There is, necessarily, a good deal of emphasis on manipulation of symbols rather than on ideas. Although the ideas used in this lesson are again used in lesson 55, they have few applications in more advanced mathematics.

We suggest that you plan to cover the lesson in one class period. If you closely follow the developmental exercises, students should not have much difficulty in understanding the ideas presented.

You may find it helpful to point out to students why we use the "base-ten expanded form" for numerals written in other bases. Also, explain that for numerals such as 376, it is understood that the base is ten. For bases other than ten, the base is indicated as part of the numeral.

The following discussion may be used following exercise I on page 231. Make the following array of x's on the chalkboard:

x x x x x
x x x x x
x x x x x
x x x x x

Say: "Suppose we want to write a numeral that expresses the number of x's shown on the board. What decimal numeral expresses this number?" [20] "What base-five numeral expresses this number?" [40_{five}] "What base-twelve numeral expresses this number?" [18_{twelve}] "Is 'Twenty equal to one eight, base twelve' a true statement?" [Yes] "Is 'Four zero, base five, is equal to twenty' a true statement?" [Yes] "Is four five and zero ones just another way of thinking of two tens and zero ones?" [Yes] "Which is the best way of thinking of the number of x's on the board?" [There is no "best" way that is mathematically defensible.] "If someone told you that he weighed three zero two, base six, pounds, could you tell immediately if he is heavier than you?" [Probably not] "Why do you suppose that almost everyone in the world now uses the decimal system?" [For easy communication and other social reasons]

After exercise T on page 232, say: "Now thinking again about the twenty x's we talked about earlier. Could you say that four zero, base five, is equal to one eight, base twelve?" [Yes]

Provision for individual differences

■ For a class of abler students, the following developmental exercises could be omitted:

A, B, page 229

G, H (column 1), B (column 2), page 230

F, H (column 1), K (column 2), page 231

With a class of slow learners, you may find it helpful to work with only one base other than base ten. Select a base (for example, base eight) and convert base-eight numerals to decimal numerals and decimal numerals to base-eight numerals. If slow learners understand the ideas involved, they will enjoy doing this work. Capitalize on this opportunity, since this activity provides students with considerable drill in computation.

Encourage slow learners to explain their work with bases to their parents. This not only will help the students to understand better what they are doing, but they also will gain a great deal of satisfaction from it. Parents who have not been exposed to this work are usually very interested in the different bases of a numeration system.

)mputing in numeration systems her than the decimal system

Objectives

The students learn to compute in numeration stems other than the decimal system.

Content overview

There is little practical value in acquiring a astery of computation in numeration systems at have bases other than ten. However, the use the properties of a place-value system when mputing in non-decimal systems leads to a bet- r understanding of the principles used when mputing with decimal numerals. For example, e use of the idea of regrouping in other bases ads to an appreciation of the "carrying" done in e decimal system.

The tables in D2, D5, and D8 are helpful in com- uting in non-decimal place-value systems. A table ould indicate all the sums or products that can e obtained by adding or multiplying numbers presented by one-digit numerals in a given base.

When making a table or performing computa- on without the aid of a table, the base of the stem that is being used must be kept in mind. o express the product $2_{three} \times 2_{three}$, for exam- le, the thinking may proceed as follows: 2×2 is our. Four is greater than the first power of the ase (in this case, three), but less than the second ower of the base (nine). Four may be regrouped n powers of three, 1 group of three (to the first ower) plus 1 one. Therefore, $2_{three} \times 2_{three} = 1_{three}$. In other words, we first think of the prod- ct expressed as a decimal numeral and then con- ert that numeral to a numeral in the base we are orking with.

One way to think about performing the computation $121_{three} \times 2_{three}$ without using a table is as follows:

Step 1: $2 \times 1 = 2$.

$$\begin{array}{r} 121_{three} \\ \times\ 2_{three} \\ \hline 2_{three} \end{array}$$

Step 2: $2 \times 2 = 11_{three}$. Write "1" and carry 1.

$$\begin{array}{r} {\scriptstyle 1} \\ 121_{three} \\ \times\ 2_{three} \\ \hline 12_{three} \end{array}$$

Step 3: $2 \times 1 = 2$. $2 + 1 = 10_{three}$.

$$\begin{array}{r} 121_{three} \\ \times\ 2_{three} \\ \hline 1012_{three} \end{array}$$

Of course, in step 2, we really think $4(3^1) = 1(3^2) + 1(3^1)$, and, in step 3, we really think $3(3^2) = 1(3^3) + 0(3^2)$. This method is somewhat sophisticated, and we suggest that you do not introduce it in the first class period spent on this lesson.

The commutative properties of the operations on natural numbers are not developed until lesson 62 of unit 6. However, the tables that are to be made for the present lesson can be used to bring out these ideas informally. For example, consider a base-five addition table:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

A diagonal in the table helps students see the commutative relationship. The numerals on one

side of the diagonal match the corresponding numerals that are on the other side; that is, the numeral in the box for (1, 4) matches the numeral in the box for (4, 1).

Remember that a commutative property is a property of an operation on a set of numbers; it is not a property of the system of numeration.

Instruction notes

■ Students will enjoy explaining to their parents why $2_{\text{three}} + 2_{\text{three}} = 11_{\text{three}}$. Encourage the students to perform mentally the computation necessary to do the regrouping when working with numerals in other bases.

After exercise C on page 233, ask: "What numbers are expressed by single digits in base six?" [The numbers from 0 through 5] "In base two?" [0 and 1] "In base twelve?" [0 through 11] "In base n?" [0 through $n - 1$]

Before exercise K on page 234, ask: "When working in base four, do you group only by fours?" [No. You group by other powers of four, too, such as sixteen and sixty-four.]

After exercise Q on page 234, say: "Think about making this base-twelve addition table. What numeral expresses the sum of 7 and 8?" [13 (read "one three")] "Do you also know what numeral expresses the sum of 8 and 7?" [Yes, 13] "If you know what numeral expresses the sum of 10 and 6, can you also use that numeral for the sum of 6 and 10?" [Yes] "By thinking in this way, you can make your table more quickly."

Before exercise J on page 236, ask: "If you know what numeral expresses the product of 8 and 4, do you also know the numeral for the product of 4 and 8?" [Yes. It is 28.]

Provision for individual differences

■ A class of abler students should complete this lesson in one day. Have them omit the following:

E, F, G, page 233
H, N, O, Q, page 234
I, J, K, page 236

You can further reduce the time needed for th lesson by having abler students complete t even-numbered "On your own" exercises. If th are interested in doing more work, however, su gest that they work subtraction examples in bas two, eight, and twelve. They might also try divide using base-eight numerals.

For classes other than abler students, devo two periods to this lesson. Complete the develo mental exercises through U on page 235 on t first day and assign "On your own" exercises through 8, 17, 19 through 24, and 29 through 3 On the second day, complete the development exercises and assign the remaining "On yo own" exercises.

Do not insist that your slow learners demo strate proficiency in computing in other bases.

5 / Special challenge / Page 237

Objectives

■ The students apply their knowledge of conver ing numerals from one base to another.

Content overview

■ In this "Special challenge," the base of the nu meration system in which the numerals are writ ten must be found. Because the digits from through 5 are used, the base must be equal to o greater than six. By converting the numerals i the second sentence of the article, the base can b found.

If the numerals in the second sentence are con sidered to be base-six numerals, they can be con verted to decimals to obtain this sentence: "H was born February 6, 1895, and died August 16 1948, at the age of 53." Because someone wh was born in 1895 and died after the anniversar of his birthday in 1948 would have died at the ag

53, it appears that the numerals actually are se-six numerals.

Other bases should be considered, too. If the merals are base-seven numerals, they can be nverted to obtain this sentence: "He was born bruary 7, 3211, and died August 18, 3434, at e age of 68." Base seven can be rejected not ly because the year 3434 has not yet occurred, t also because someone who was born in 3211 d died in 3434 would have died at the age of 3, not 68. Other bases may be tried, but a true tement is expressed by the second sentence ly when the numerals are considered to be se-six numerals.

When it has been determined that the numerals e base-six numerals, it is relatively simple to nvert all the numerals in the article to decimal merals.

---

ecimal numerals for numbers reater than one thousand

Objectives

The students learn to read and write numerals r numbers greater than 1000. They learn to ex- ress some numbers by scientific notation.

Content overview

This lesson deals with numerals of the kind ncountered in science and in certain every- lay situations. For example: The Canadian ational debt of over $17,000,000,000 is fre- uently mentioned in newspapers. A light-year 5,880,000,000,000 statute miles, or the distance raversed by light in one year) is introduced in unior high school science classes as a unit of neasure in expressing stellar distances. The gross ational product in Canada is valued at over $36,000,000,000 per year. The population of the world is more than 3,000,000,000 persons. In one year the production of corn in the United States was 4,352,668,000 bushels. Normally, such numbers as the number of molecules in a glass of water and the number of electrons in the universe are expressed in scientific notation. For example, the estimate of the diameter of the universe is expressed as 10^{24} kilometres.

In reading a numeral, we read the digits in each period and then the name of that period (beginning with the period farthest to the left). The period farthest to the right is sometimes called the ones' period or the units' period. No name is used for this period when reading a numeral.

Names of periods that are not mentioned in the text and the number of places to the right of each period (to the decimal point) are as follows: The period next greater than quadrillion is quintillion (18); then sextillion (21), septillion (24), octillion (27), nonillion (30), decillion (33); undecillion (36), duodecillion (39), tredecillion (42), quattuordecillion (45), quindecillion (48), sexdecillion (51), septendecillion (54), octodecillion (57), novemdecillion (60), and vigintillion (63).

Your students may be interested in hearing about a "googol," that is, the number named by the numeral 10^{100}, or a "1" followed by one hundred zeros. The name "googol" is reputed to have been thought of by a small boy. Note that googol is not the name of a period.

A "googolplex" is the number named by the numeral $10^{10^{100}}$, or a "1" followed by a googol of zeros.

Scientific notation is simply a concise method of expressing numbers. For example, the number 60,000,000,000,000 can be expressed in scientific notation by the numeral 6×10^{13}. In *STM 2* we show that a number like 435,000,000,000 can be expressed by the numeral 4.35×10^{11}.

Instruction notes

■ The content of this lesson not only has practical values, but also will be of great interest to your students.

Begin the lesson by writing this numeral on the board: 5,880,000,000,000. Ask: "Who thinks he can read this numeral?" [Let a few try until one does it correctly. It is 5 trillion, 880 billion.] "This numeral is not one that I picked at random. It expresses the number of miles that light travels in one year. This number of miles is called a light-year. Astronomers use it as a unit of distance. Can you think of other examples of the use of very large numbers?"

As your students read the numerals in the first block of exercises (H through M on page 238), caution them not to use the word "and" between periods. Remind them that they should use "and" only to indicate a decimal point when reading a numeral.

After exercise F on page 239, say: "The national debt of Canada is about 17 billion dollars. What numeral expresses 17 billion?" [17,000,000,000] Write this numeral on the chalkboard. "How is 17 billion expressed in scientific notation?" [17×10^9]

Provision for individual differences

■ Have the abler students collect numerals that have many digits from science books and magazines. They could also find and list the names of the periods that are greater than those given in the lesson.

The abler students might consider the magnitude of the number named by 9^{9^9}. Notice that the exponent is 9^9, and $9^9 = 387{,}420{,}489$.

Have all the students collect numerals with many digits from any source—newspapers, magazines, and books. Display their collections on the bulletin board. Have your slow learners practice oral reading of the numerals that are displayed on the bulletin board.

5 / Special challenge / Page 241

Objectives

■ The students use what they have learned abou the properties of a modern numeration system.

Content overview

■ A base-seven numeration system is used in th first three exercises. Seven words (or digits) fo counting numbers are provided: "re" names 1 "mi" names 2, "fa" names 3, "sol" names 4, an so on. "Re-do" names 10_{seven}, which means tha "do" names 0.

There are several systematic ways of findin correct responses to the exercises. One way is de scribed below.

1 Make a list of the names of the first seve natural numbers and match a base-seven numera with each word, as follows:

do	re	mi	fa	sol	la	ti
0	1	2	3	4	5	6

2 Find a base-seven numeral for a given decima numeral. For example, a base-seven numeral fo 45 is 63_{seven}.

3 Use the list that is provided for exercise 1 to replace each digit by a written sound. For example, replace 6 by "ti" and 3 by "fa" in 63_{seven} to obtain "ti-fa."

6 / 57 Exploring ideas / Pages 245-249

Equivalent sets

Objectives

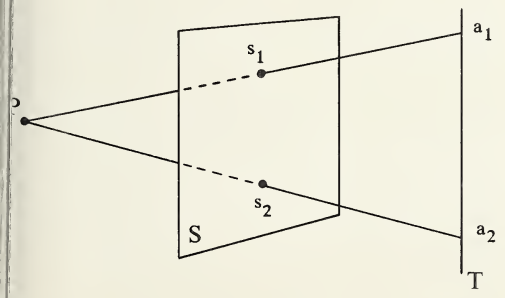
■ The students learn what is meant by one-to-one mapping, one-to-one correspondence, and equivalent sets.

Content overview

The basic objectives of unit 6 are to develop the ·operties of natural numbers and to define a ımber system. The mathematical concepts pre- nted in this unit are very important because ey form a foundation upon which other mathe- atical concepts are developed.

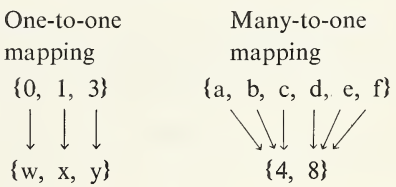
The meaning of natural numbers and the order- g of these numbers depend on the idea of one-)-one correspondence. To make possible a defi- ition of one-to-one correspondence that is clear, ccurate, and complete, we consider the idea of apping. A mapping is a special type of pairing ıe members of one set with the members of an- ther set.

On a geographical map, each point is uniquely aired with a specific point on the surface of the arth. Using the diagram below, the process of aking a map may be described as follows: oint P represents a source of light from which ght rays pass through a filmstrip S to a screen T. oints on the filmstrip are mapped onto points n the screen. The points on the screen form a ap of the points on the filmstrip.

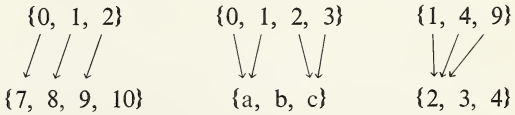


This idea of mapping has been extended by nathematicians. For example, if any two sets, A ınd B, are given, and if we associate each member)f A with a member of B, and if each member of B is used, we have a mapping of A onto B. If each member of set A has an image in set B, and each member of B is used just once, we have a one-to-one mapping. Whenever more than one member of a given set is mapped onto a member of a second set, we refer to the mapping as many-to-one. In a many-to-one mapping, a member of the second set is used more than one time.

Although there are *into* mappings and *onto* mappings, we speak only of *onto* mappings in *STM 1*. Actually, all mappings are "into" mappings. An "onto" mapping is a special case of an "into" mapping. For "onto" mappings, all the members of the second set are used, while for "into" mappings, all the members of the second set may or may not be used. In an "onto" mapping, each member of the second set is the *image* of at least one member of the first set, as shown in the diagrams below:



Three examples of mappings that are "into," but not "onto," are represented below:

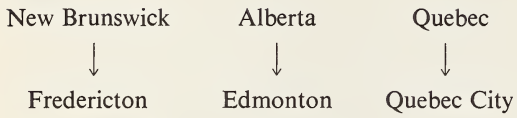


Note that the arrows in these diagrams show which member of one set is mapped onto (or into) the member of the second set.

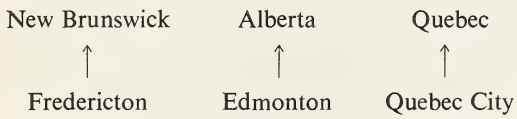
Two one-to-one mappings are necessary to make a one-to-one correspondence. Consider sets C and D.

C = {New Brunswick, Alberta, Quebec}
D = {Fredericton, Edmonton, Quebec City}

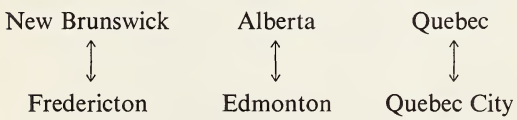
First, each province is mapped onto its capital city.



Next, each capital city is mapped onto its province.



Notice that, in both mappings, the same members of each set are linked together. In a one-to-one correspondence, the "matching member" of a particular member of either set is uniquely determined by either of the two mappings. The two mappings above form a one-to-one correspondence. A one-to-one correspondence, made up of two mappings combined, is usually described in a diagram by means of double-headed arrows.



The two sets, C and D, are in one-to-one correspondence because there is a one-to-one mapping of the members of set C onto the members of set D, and there is a one-to-one mapping of the members of set D onto the members of set C. Any two sets that can be put in one-to-one correspondence are *equivalent sets*.

Since {oriole, robin} and {elm, oak} can be put in one-to-one correspondence, they are equivalent sets. Moreover, any set that can be put in one-to-one correspondence with {elm, oak} is equivalent to {elm, oak}. In lesson 58, we use the idea of equivalent sets to associate natural numbers with sets. All finite sets that are equivalent to each other are said to be associated with the sam natural number. Therefore, a natural number is common property of equivalent sets. The natur number of a set that is not infinite tells how man members are in the set.

Instruction notes

■ Use the displays in the lesson to explain "map ping." Make sure the students understand tha in the mapping of one set onto another, eac member of both sets must be included, and tha two one-to-one mappings are needed to form one-to-one correspondence.

There are many examples of mappings an one-to-one correspondences in the experience o the students. They may be able to suggest some To start a discussion, you might ask how one ca determine whether or not there are enough text books for students without counting either text books or students. Or you might ask how the can show, without counting, that they have th same number of fingers on each hand.

One-to-many mappings and many-to-one map pings can be explained by using situations tha arise naturally in school experiences. Consider for example, an overcrowded school in which theoretically, each locker is "mapped onto" a stu dent. If the available lockers are assigned before all students receive an assignment, some of the lockers must be mapped onto two students. Note that each member of the set of lockers is mapped onto at least one member of the set of students. Also, each student is used in the assignment (mapping) of the lockers to the students. Such a mapping is a one-to-many mapping.

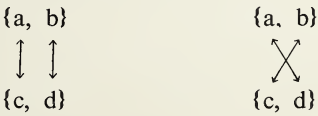
Now, suppose that, before school opens, a clerk is engaged to assign a number to each locker. The mapping that results is a one-to-one mapping of the set of numbers onto the set of lockers. This mapping automatically determines another mapping, a one-to-one mapping of the set of lockers onto the set of numbers. These two

ppings show that there is a one-to-one corre-
ondence between the set of lockers and the set
numbers. Have students draw diagrams to
ow a one-to-one mapping of lockers onto num-
rs, a one-to-one mapping of numbers onto
kers, and a one-to-one correspondence be-
een the lockers and the numbers. The dia-
ms the students draw might look like these:

apping of kers onto mbers	Mapping of numbers onto lockers	One-to-one correspondence
⟶1	L_1⟵1	L_1⟷1
⟶2	L_2⟵2	L_2⟷2
⟶3	L_3⟵3	L_3⟷3

Make sure the students understand that in a
e-to-one correspondence, the image of a par-
ular member of either set is uniquely deter-
ned by either of the mappings from which the
rrespondence is structured. For example, 1 is
apped onto L_1, and L_1 is mapped onto 1.

The students should remember that the order
which the members of a set are listed does not
ange the set. Thus, different mappings can re-
lt when mapping one set onto another set.
ence, if two equivalent sets are given, there are
ually several one-to-one correspondences be-
een the two sets. If M = {a, b} and N = {c, d},
en there are two possible one-to-one corre-
ondences, as shown below:



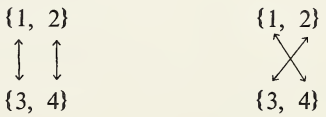
Discourage students from using the process of
unting to determine the equivalence of sets, be-
use the meaning of the numbers and the possi-
lity of counting depend on one-to-one corre-
ondence and equivalent sets. Students will learn
subsequent lessons how natural numbers are
associated with equivalent sets, and how this
leads to the ordering of the natural numbers.

Provision for individual differences

■ For a class of slow learners, draw from the experiences of the students to set up mappings and one-to-one correspondences. You can use sets of students, sets of chairs, and sets made up of pieces of chalk, erasers, pencils, and so on.

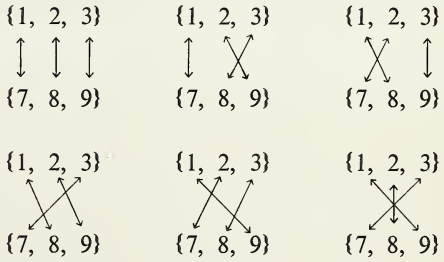
Abler students could consider the following:

1 A = {1, 2} and B = {3, 4}. Make a diagram to represent all the possible ways in which A and B can be put in one-to-one correspondence. [The diagrams below show all possible correspondences.]



2 In exercise 1, how many one-to-one correspondences of A and B are there? [2 × 1, or 2] Of B and A? [2 × 1, or 2]

3 C = {1, 2, 3} and D = {7, 8, 9}. Make a diagram to represent all the possible ways in which C and D can be put in one-to-one correspondence. [The diagrams below show all possible correspondences.]



4 In exercise 3, how many one-to-one correspondences of C and D are there? [3 × 2 × 1, or 6] Of D and C? [3 × 2 × 1, or 6]

5 Sets E and F each have four members. How many one-to-one correspondences of E and F are there? [$4 \times 3 \times 2 \times 1$, or 24]

6 Can you determine the number of ways in which two equivalent sets can be put in one-to-one correspondence without drawing a diagram or listing each way? [Consider set A and set B as equivalent sets, each having three members. There are three members of B onto which the first member of A can be mapped. After the first member of A is mapped, there are two members of B onto which the second member of A can be mapped. After the first two members of A are mapped, there is just one member of B onto which the third member of A can be mapped. Therefore, the number of possible mappings is $3 \times 2 \times 1$. The number expressed is called "3 factorial" or "factorial 3" and is symbolized by "3!" when thought of in this way.]

7 Set G has four members and H has three members. How many one-to-one mappings of G onto H are possible? [None]

8 Sets I and J are equivalent. J has two members. How many one-to-one mappings of I onto J are possible? [2×1, or 2]

9 Set K has three members. There are how many possible one-to-one mappings of K onto K? [$3 \times 2 \times 1$, or 6]

Abler students may also find it interesting to consider one-to-one correspondence and equivalence for infinite sets. In lesson 59 we give a precise meaning to the idea of a proper subset by using the idea of mapping, and we then use the idea of proper subset to order the natural numbers. Two infinite sets are equivalent if they can be put in one-to-one correspondence. This holds true even when one infinite set is a proper subset of the other infinite set. In fact, some mathematicians define an infinite set as a set that can be put in one-to-one correspondence with a proper subset of itself.

It is not possible to determine directly wheth or not two infinite sets can be put in one-to-o correspondence. Mathematicians do so indirect They find a rule whereby each member of one i finite set can be paired uniquely with a member another set.

The diagram below represents one way which the set of natural numbers and the set even numbers can be put in one-to-one corr spondence. U = N.

$$\begin{array}{c} \{0,\ 1,\ 2,\ \ldots,\ n,\ \ldots\} \\ \updownarrow\ \updownarrow\ \updownarrow\ \quad\quad \searrow \\ \{0,\ 2,\ 4,\ \ldots,\ 2n,\ \ldots\} \end{array}$$

The diagram below represents one way which the set of natural numbers and the set odd numbers can be put in one-to-one corr spondence. U = N.

$$\begin{array}{c} \{0,\ 1,\ 2,\ \ldots,\ n,\ \ldots\} \\ \updownarrow\ \updownarrow\ \updownarrow\ \quad\quad \searrow \\ \{1,\ 3,\ 5,\ \ldots,\ 2n+1,\ \ldots\} \end{array}$$

If we can determine a rule that enables us t put two infinite sets in one-to-one correspond ence, we know that the sets are equivalent.

---

6 / Keeping skillful / Page 249

■ The "Keeping skillful" exercises may be use to help maintain understanding of mathematica ideas and to maintain competence in basic skills The exercises in each "Keeping skillful" sectio review ideas and computation that have been pre sented in previous lessons.

Assign as work to be done outside of class the ercises that you think should be completed by ur average and abler students. Discuss in class ly those exercises that caused difficulty.

For slow learners, provide some class time for ork on the exercises, and give as much individ- l help as possible. The "Keeping skillful" exer- ses are especially useful in helping slow learners master fundamental ideas and develop greater ill in computation.

58 Exploring ideas / Pages 249-251

atural numbers as properties of sets

Objectives

The students learn how to associate a natural umber with a standard set and with each set uivalent to a standard set.

Content overview

This lesson shows how the natural numbers nerge in a special way from equivalent sets. We gin with the empty set, symbolized by { }. We ssociate the natural number zero with the empty t. The standard set that is associated with the atural number 1 is {0}. Now we have the natural umbers 0 and 1 to work with, and we can make nother standard set, {0, 1}. This standard set is ssociated with the natural number 2.

Each standard set is constructed out of the atural numbers previously defined. Finally, we an generalize by saying that if m is replaced by ny member of {1, 2, 3, . . .}, then m equals the umber of members in {0, 1, 2, . . ., $m-1$}.

Many of us find it difficult to think precisely of n idea as basic as the natural number one, or ve, or forty-six. Now that we have an intuitive nderstanding of sets, we can think of natural umbers in terms of sets. The number 0 is a common property of the empty set and all sets equivalent to the empty set. The number 1 is a common property of {0} and all sets equivalent to {0}. The number 2 is a common property of {0, 1} and all sets equivalent to {0, 1}.

Note that it is not difficult to think of sets that are equivalent to standard sets. For example, {father, mother} is equivalent to {0, 1}. Other sets equivalent to {0, 1} are {//}, {table, chair}, and {bat, ball}.

Instruction notes

■ It is important for students to understand that a natural number is a common property of a standard set and all sets that are equivalent to the standard set. For example, the natural number 4 is a common property of {☆, #, ?, []} and the standard set {0, 1, 2, 3}. These sets have a common property because they can be put in one-to-one correspondence and thus are equivalent. The set {quart, pint, gallon, litre}, as well as infinitely many other sets, has this common property, because it is also equivalent to {0, 1, 2, 3}. Moreover, a set that is not equivalent to {0, 1, 2, 3} does not have this common property.

Your students should be able to make comparable observations concerning any other member of the set of natural numbers. Each is a common property of a standard set and all sets equivalent to the standard set.

To motivate the study of this lesson, begin by asking, "Who thinks he can give a good definition for the number zero?" [Let a few students try. They probably will use everyday terms such as "not any," "none," and so on.] "Can anyone think of mathematical ideas that can be used to define the number zero? In this lesson, you will use ideas about sets to describe the number zero and other natural numbers."

Before exercise K on page 250, ask: "Is each member of a standard set a natural number?"

[Yes] "What did we have to explain before we could use the set whose member is zero?" [What is meant by 0]

After exercise M on page 251, hold up four pieces of chalk and say: "Think about the set whose members are the pieces of chalk in my hand. What number is a property of this set?" [4] Hold up four pencils and ask a similar question. Then ask: "How many sets have the property of the natural number four?" [Infinitely many] "How many sets have the property of eight?" [Infinitely many] "How many sets have the property of zero?" [Only one: the empty set]

Provision for individual differences

■ For a class of slow learners, omit exercise 19 in the "On your own" section and the following developmental exercises:

D, page 250
T, U, page 251

---

6 / Special challenge / Page 252

Objectives

■ The students use what they have learned about one-to-one correspondence and equivalent sets to understand the concept of a denumerable set.

Content overview

■ Any set that is equivalent to the set of natural numbers is a denumerable set. Because N is an infinite set, any denumerable set is also an infinite set. Not all infinite sets are denumerable, however. To know that a given infinite set is denumerable, it must be demonstrated that the given set and N can be put in one-to-one correspondence. Therefore, a rule must be found to order the members of the given set.

We can show that the set of even numbers is denumerable by using the following diagram:

$$\begin{array}{l} \{0,\ 1,\ 2,\ \ldots,\ n,\ \ldots\} \\ \updownarrow\ \updownarrow\ \updownarrow\ \qquad \updownarrow \\ \{0,\ 2,\ 4,\ \ldots,\ 2n,\ \ldots\} \end{array}$$

We can show that the set of odd numbers is d numerable by using this diagram:

$$\begin{array}{l} \{0,\ 1,\ 2,\ \ldots,\ n,\ \ldots\} \\ \updownarrow\ \updownarrow\ \updownarrow\ \qquad \searrow \\ \{1,\ 3,\ 5,\ \ldots,\ 2n+1,\ \ldots\} \end{array}$$

Set T, tabulated in the display in the student book, is denumerable because its equivalen with N can be demonstrated by the followi diagram:

$$\begin{array}{l} \{0,\ 1,\ 2,\ \ldots,\ n,\ \ldots\} \\ \updownarrow\ \updownarrow\ \searrow\ \qquad \searrow \\ \{1,\ 10^1,\ 10^2,\ \ldots,\ 10^n,\ \ldots\} \end{array}$$

The equivalence of set S and set N can b demonstrated by this diagram:

$$\begin{array}{l} \{0,\ 1,\ 2,\ \ldots,\ n,\ \ldots\} \\ \updownarrow\ \updownarrow\ \updownarrow\ \qquad \updownarrow \\ \{0,\ 1,\ 4,\ \ldots,\ n^2,\ \ldots\} \end{array}$$

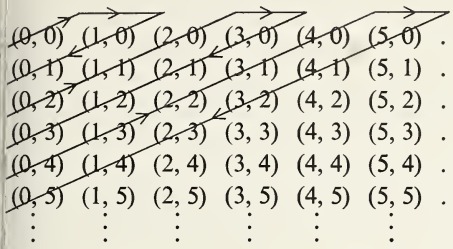
Set V is not denumerable, because it is a finit set.

We can use this diagram to show that set C i the display is denumerable:

$$\begin{array}{l} \{0,\ 1,\ 2,\ \ldots,\ n,\ \ldots\} \\ \updownarrow\ \updownarrow\ \updownarrow\ \qquad \searrow \\ \{1,\ 2,\ 3,\ \ldots,\ n+1,\ \ldots\} \end{array}$$

To show that N × N is denumerable, a some what different procedure must be used. The char

low shows one way to arrange, or order, the embers of N × N.



By ordering in this way, we can find the mem- r of N × N that is matched with any given ember of N. As an example, 0 is matched with , 0); 1 is matched with (1, 0); 2 is matched with , 1); and so on.

The set of natural numbers is denumerable be- use it can be put in one-to-one correspondence ith itself.

Instruction notes

The eleven sets of "Special challenge" exercises *STM 1* may be used to provide enrichment for bler students. The suggestions below apply in eneral to each of these sets of exercises. Instruc- on notes are given for a specific "Special chal- nge" when such comments are needed.

Have abler students work on each "Special allenge" independently and without prior dis- ussion. Encourage these students to try to an- wer the exercises themselves and to develop a ethod of explaining the solution they obtain.

Remember that the answers for the "Special hallenge" exercises are given in the pages at the ack of the *Teaching Guide*. They are not in- cluded in the *Response Verification Books* for *TM 1*. When abler students complete a "Special hallenge," either have them check their own an- wers or discuss their answers with them while he other students in the class are working inde- endently.

6 / 59 Exploring ideas / Pages 252-255

Ordering of the natural numbers

Objectives

■ The students learn that the set of natural numbers has an order property and a successor property.

Content overview

■ This lesson develops some of the properties of natural numbers. One of these is the *order* property. The order relations that are relevant to the natural numbers are "less than" and "greater than." Either of these may be selected as a point of departure, since, for every replacement of the variables, when the universe is N, the statement obtained from "$a < b$" gives the same information as the statement obtained from "$b > a$." We use "less than" in the lesson.

To order the natural numbers, we must establish criteria for determining (1) whether or not a is less than b for any given replacement for the variables, (2) given a, what number should immediately follow a, and (3) which member of N shall be first in the ordered set. We cannot use counting to order the natural numbers because the process used to count is based on the assumption that the natural numbers have already been ordered. We use the ideas of standard set and proper subset to order the natural numbers.

The idea of proper subset extends the idea of subset. When a subset does not contain all the members of a set, that subset is a *proper subset*. Remember that the empty set is, by convention, a proper subset of every set except itself.

Consider the number 4 and the number 7. The standard set of 4 is {0, 1, 2, 3}, and the standard set of 7 is {0, 1, 2, 3, 4, 5, 6}. Since the stand-

ard set of 4 is a proper subset of the standard set of 7, the number 4 is less than the number 7. In general, when the standard set of one number is a proper subset of the standard set of another number, the first number is less than the second.

Now we can determine the number that immediately follows 4. Each member of the standard set of 4 is less than 4. Consider the set that contains the number 4 and the members of the standard set of 4. This set is {0, 1, 2, 3, 4}, the standard set of 5. We select this number, 5, as the number that will immediately follow 4, and we refer to 5 as the *successor* of 4. In the same way, we can identify the successor of any natural number, and we say that the natural numbers have the *successor property*. We can formalize this by saying that the successor of x is $x+1$, when U = N. Every member of the set of natural numbers has exactly one successor. Note, however, that zero is not the successor of a member of N. Consider a set of natural numbers with these properties: (1) 0 is a member of this set, and (2) if x belongs to this set, the successor of x also belongs to this set. We know that this set contains all the natural numbers.

We can now arrange the natural numbers in a special order so that each natural number will be less than any other natural number following it. In this way, we obtain the ordered set of natural numbers with 0 as the first, since 0 is the least natural number; then the natural numbers 1, 2, 3, and so on, follow.

After we have ordered the set of natural numbers, another idea emerges. It is possible to determine how many numbers are between any two natural numbers in the ordered set N and to determine what these numbers are by using a compound condition and finding its solution set. If a and b are variables for two members of N, and x is a variable for a member between a and b, we can determine the number of members between a and b by determining the solution set of th condition $a < x \wedge x < b$. The only restriction that a must be less than b. For example, if we r place a by 11 and b by 15, we obtain the senten $11 < x \wedge x < 15$. Since the solution set of th condition is {12, 13, 14}, we know that there ar three members between 11 and 15 in the ordere set of natural numbers and that these numbe are 12, 13, and 14.

Once we have the order property, we can assig numbers to some of the points in the number lin On a picture of the number line, the dot farthe to the left is assigned the number zero. Dots pic turing the points assigned to the natural numbe are equally spaced to the right of the dot for zer For each point a, there is a unique point assigne to the successor of a, or $a+1$.

Instruction notes

■ Spend enough time on this lesson so that stu dents understand that a set of natural numbers i ordered if, for any two members, a and b, ther is a way to determine that a precedes b or that follows b, but that a cannot both follow and pre cede b. There are many examples of ordered set that you can use in a class discussion: the days o the week, the notes of a musical scale, places on longitudinal meridian ordered according to direc tion *to the north of* or *to the south of* a startin point, and so on. Have the students select ex amples of sets that can be ordered and investigat to see if they exhibit the same properties as th ordered set of natural numbers.

To introduce the lesson, say: "You used nat ural numbers long ago, but the last lesson helpe you find a more precise way to think about these numbers. It is possible that you may be familia with the properties you are going to study in thi lesson. However, we need to study these proper ties to have a basis for future work."

Before exercise N on page 253, ask: "Is an or dering of the natural numbers indicated in D2?"

Yes] "Is the idea of 'less than' for natural numbers used in this ordering?" [Yes]

After exercise L on page 254, say: "Let us develop a rule to find how many natural numbers are between two given natural numbers. You know that there are 6 natural numbers between 4 and 11. Is 6 the difference of 11 and 4?" [No. Six is one less than the difference.] "You know that there are no natural numbers between 16 and 17. Is 0 the difference of 17 and 16?" [No. Zero is one less.] "There are 7 natural numbers between 472 and 480. Is 7 the difference of 480 and 472?" [No. Seven is one less.] "Without counting, how can you find how many natural numbers are between two given natural numbers?" [By subtracting 1 from the difference of the two given numbers]

Provision for individual differences

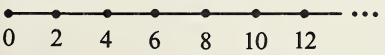
Abler students who quickly master the content of the lesson could identify order relations other than "less than" and apply the principles learned to the ordering of sets other than the natural numbers. These students could also consider the exercises below.

For exercises 1 through 6, E is the set of even numbers.

1 Why is 0 a member of E? [An even number is a number that has 2 as a factor. Since $2 \times 0 = 0$, 0 is a factor of 0.]

2 Tabulate E. [{0, 2, 4, . . .}]

3 Make a picture of a number line that indicates the order of the first seven members of E.



4 In E, what is the successor of 16? [18]

5 U = E. What is the successor of y? [$y + 2$]

6 Tabulate the members of E between 26 and 34. [{28, 30, 32}]

For exercises 7 and 8, W is the set of natural numbers less than 8.

7 Does W have the successor property? [No. There is no successor for 7.]

8 Does W have the betweenness property? [Yes]

9 Use the relation "east of" to order the members of {Saskatchewan, Newfoundland, British Columbia, New Brunswick, Ontario}. [{Newfoundland, New Brunswick, Ontario, Saskatchewan, British Columbia}]

10 Use the relation "older than" to order the members of {teen-ager, adult, baby, child, toddler}. [{adult, teen-ager, child, toddler, baby}].

11 Use the relation "alphabetically precedes" to order the members of the set of words {Charlie, baker, able, dog}. [{able, baker, Charlie, dog}]

For slow learners who need further practice, provide exercises similar to "On your own" exercises 14 through 17 and 21 through 24.

---

6 / Checking up / Page 255

End-of-block tests

■ The end-of-block tests are designed to test the students' ability to use the mathematical skills developed in the lessons that immediately precede the tests; the tests cover from three through seven lessons. These tests are not designed to test the students' ability to verbalize.

See pages 54 and 55 of this *Teaching Guide* for a detailed discussion of the end-of-block tests and how to conduct an effective testing program.

---

6 / Applying mathematics / Page 256

■ Throughout *STM 1*, maintenance in problem solving is provided by sets of problems that ap-

pear under the heading "Applying mathematics." These sets of problems provide a cumulative review of various kinds of problems and problem-solving procedures.

See pages 83 and 84 of this *Teaching Guide* for suggestions for using the "Applying mathematics" problems. See pages 129 and 132-133 of this *Teaching Guide* for a discussion of social and business terms that are introduced in other sets of "Applying mathematics" problems.

---

6 / 60 Exploring ideas / Pages 256-260

The closure property of addition

Objectives

■ The students study the meaning of the sum and of the difference of two natural numbers, binary operations, and the closure property of addition.

Content overview

■ This lesson gives mathematical meaning to both the operation of addition of natural numbers and the closure property of this operation. The relation of subtraction to addition is explained in the latter part of the lesson. In the course of these developments, the important basic ideas of sets are used.

An understanding of the operation of addition of two natural numbers involves an understanding of the sum of two natural numbers. The *sum of two natural numbers* is defined as the number associated with the set that is the union of two disjoint sets. Suppose that A and B are disjoint sets. For natural numbers, if a is associated with set A, and b is associated with set B, then the sum $a + b$ is associated with $A \cup B$.

The sum of 1 and 5 is indicated by $1 + 5$. Suppose that A = {cup} and B = {bowl, saucer, plate, glass, pitcher}. Then $1 = n(A)$ and $5 = n(B)$. By definition, $1 + 5 = n(A \cup B)$. Since $A \cup B$ is equivalent to the standard set of the natural number 6, then $6 = n(A \cup B)$. It follows, therefore, that the standard name for the sum $1 + 5$ is the numeral 6.

The number associated with $A \cup B$ is expressed differently when it is thought of as being associated with $B \cup A$. For the sets described in the preceding paragraph, $n(A \cup B) = 1 + 5$ and $n(B \cup A) = 5 + 1$. From examples like this, the commutative property of addition can be generalized: $a + b = b + a$. The commutative properties of multiplication and addition of natural numbers are presented in lesson 62.

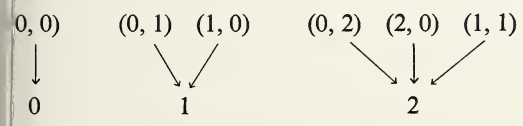
An operation on a set is a rule that assigns to each member (or each group of members) in that set a uniquely defined member in the same set. "Finding the successor of a member," for example, is an operation useful in the ordering of a set. If we are ordering the set of natural numbers, we know that, if a is a member of N, $a + 1$ is a description of the successor of a for every permissible replacement of a. "Finding the successor" is an operation on a single member of a set.

The majority of operations in arithmetic are binary operations. A binary operation on a set is a rule that assigns to each *pair* of members of the set a uniquely defined member in the same set, or, in other words, a mapping (following some rule) of each ordered pair of the Cartesian set formed from the given set onto a unique object of the given set. The operation of addition of natural numbers is a binary operation.

Binary operations can best be defined and understood in terms of ordered pairs because ordered pairs occur frequently in everyday experience. Time is one of many determinants of order. Suppose that you deposited x dollars in your bank account on Thursday, and on the following Tuesday you made a deposit of y dollars. There is an ordered pair, (x, y), that uniquely deter-

ines a number $x + y$ by means of a particular inary operation called addition for all permissile replacements of the variables. If x and y are atural numbers, $x + y$ is also a natural number. f we replace (x, y) by $(25, 48)$, then $x + y$ becomes $25 + 48$, or 73. We say that, for each x and , $x + y$ is uniquely determined.

Addition is a special kind of mapping. Recall hat, whenever more than one member of a given et is mapped onto a single member of a second et, we refer to the mapping as many-to-one. Addition is really a many-to-one mapping of the Cartesian set (all possible ordered pairs) of natural numbers onto the set of natural numbers. The ordered pairs mapped under addition onto he first three members of N are shown below:



Notice that only one member of $N \times N$ maps onto 0 by addition. In all other cases, the mapping is many-to-one.

Since addition is an operation that maps each pair of elements in N onto an element that is also n N, we say that the set of natural numbers is *closed under addition*.

Some of the subsets of N are not closed under addition. One of these is {1, 3, 5, . . .}. Since addition maps (1, 3) onto 4, and 4 is not an odd number, we know that {1, 3, 5, . . .} is not closed under addition. {1, 3, 5, . . .} can be closed under the operation of "finding the successor," if we define the successor of an odd number a as $a + 2$ for each permissible replacement of a. Notice that this operation is not binary.

We find that N is not closed under subtraction. If we select an ordered pair of natural numbers such as (7, 2), and we define the natural number that satisfies $7 = x + 2$ as the difference of 7 and 2, the ordered pair (7, 2) is mapped onto 5, or $7 - 2$, because $7 = 5 + 2$. However, the condition $8 = x + 11$ cannot be satisfied by a member of N because the ordered pair (8, 11) cannot be mapped under subtraction onto a member of N. Only those ordered pairs in $N \times N$ whose first component is greater than or equal to the second component can be mapped by subtraction onto a given member of N.

The number of ordered pairs mapped under subtraction onto any given member of N is infinite. Each member of {(0, 0), (1, 1), (2, 2), . . .} is mapped onto 0; each member of {(2, 0), (3, 1), (4, 2), . . .} is mapped onto 2; and so on. But such pairs as (0, 1), (1, 2), and (5, 9) are not included in the mapping. Subtraction, then, is a many-to-one mapping of a subset of $N \times N$ onto N. Set N is not closed under subtraction because this operation may not assign to two members of N a uniquely defined element in N.

Notice that subtraction is given meaning in the lesson in terms of addition. We say that for each true statement obtained from $x - y = z$, a true statement is also obtained from $x = y + z$. For each variable, U = N.

This lesson contributes to the understanding of what is meant by a number system. Turn to lesson 66 to see how the ideas developed in this lesson are combined with other ideas to develop the notion of the natural-number system.

Instruction notes

■ Because the ideas in this lesson are basically important, and because some of the ideas are completely new to the students, follow the developmental exercises closely in presenting the lesson. Explain that, although they already know how to compute using addition, this lesson will help them understand how sets can be used to develop a mathematical basis for addition.

Plan to discuss each of the developmental exercises in class, but if students appear to be confused with some of the difficult ideas presented in the lesson, take time to expand on the discussion and provide additional explanatory material or more examples. For example, some students may find the concept of closure particularly difficult.

Provision for individual differences

■ For slow learners plan to devote at least two class periods to this lesson. You might stop with the definition of closure on page 259 on the first day and assign the first fifteen "On your own" exercises. On the second day, finish the lesson.

Slow learners may gain new insight concerning the basic facts of addition by continuing the mapping of ordered pairs mapped under addition onto the first three members of N discussed in the "Content overview" for this lesson.

Slow learners will be able to accept and apply the concepts developed in this lesson, but you can expect them to have difficulty in verbalizing the ideas. Your deliberate, precise use of the correct terms in class discussion will help them in this respect.

Abler students could consider the following:

1 You know that addition is an operation on numbers. What two operations on sets have you studied earlier? [Union and intersection]

2 Suppose that set A is the set of natural numbers less than 100. Is A closed under addition? Give an example to verify your answer. [No, because $1 + 99 = 100$, and 100 is not a member of A.]

3 What natural number is the first component of the ordered pair whose second component is 20 when the ordered pair is mapped onto 30 by addition? [10]

4 By addition, each member of $C \times C$ is mapped onto a member of what set? [{2, 3, 4, . . .}]

5 By what natural number should you replace x so that (x, x^2) is mapped onto 30 by addition? [5]

The closure property of multiplication

Objectives

■ The students learn what is meant by the product and the quotient of two natural numbers. They also learn that multiplication of natural numbers is a binary operation that has closure.

Content overview

■ In *STM 1*, addition is defined by means of the union of two disjoint sets, and multiplication is defined by means of the Cartesian set of two given sets that may or may not be disjoint. The number onto which two numbers are mapped by addition is called their sum. The number onto which two numbers are mapped by multiplication is called their product. Both addition and multiplication are many-to-one mappings of members of $N \times N$ onto N.

Suppose that $A = \{0, 1\}$ and $B = \{0, 1, 2, 3\}$. 2 is associated with A, and 4 is associated with B. Then 2×4 is associated with $A \times B$. To verify, we find that $A \times B = \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3)\}$. Since $A \times B$ is equivalent to the standard set $\{0, 1, 2, 3, 4, 5, 6, 7\}$, we know that the product 2×4 is the natural number 8. Notice that sets A and B do not have to be disjoint.

Multiplication of natural numbers, like addition, is a binary operation. It is a mapping of the members of $N \times N$ onto the members of N. If z is the product of x and y, then (x, y) maps onto z by using the operation of multiplication.

A given natural number is associated with a finite set. The Cartesian set generated from any two finite sets is also a finite set. Because each product of natural numbers is associated with a

inite Cartesian set, the set of natural numbers is losed under the operation of multiplication; that s, there always *is* a product that is a member of he set.

In *STM 1*, we define subtraction in terms of ıddition, and we define division in terms of multi-)lication. Consider the ordered pair (10, 5). We lefine the natural number that satisfies the condi-ion $10 = 5x$ as the quotient of 10 and 5. (10, 5) is napped onto 2, or $10 \div 5$, by division because $10 = 5(2)$ is a true statement. However, the condi-ion $6 = 12x$ cannot be satisfied by a member of N, so (6, 12) cannot be mapped by division onto ı member of N. Similarly, the condition $6 = 0 \cdot x$ cannot be satisfied by a member of N, so (6, 0) cannot be mapped by division onto a member of N. Thus, we conclude that N is not closed under division.

The two special cases that arise in connection with the attempt to divide by zero are developed on page 292. Division of a counting number by zero is meaningless because $a \div 0 = x$ means $a = 0 \cdot x$. Since $0 \cdot x = 0$ for every replacement of the variable, $a \div 0 = x$ has no solution, when $U = C$. Division of zero by zero is different. We say $0 \div 0$ is indeterminate because $0 \div 0 = x$ means $0 = 0 \cdot x$. Since every natural number satisfies $0 = 0 \cdot x$, $0 \div 0$ may be any natural number. Thus, we say that we cannot divide by zero.

Instruction notes

■ The ideas developed in this lesson and in the previous lesson have much in common. The similarity of the lessons should help to clarify the thinking of the students and should strengthen their understanding of the concepts presented in the two lessons.

As with the previous lesson, you should closely follow the developmental exercises in presenting the lesson.

If students raise the question of division by zero and find it difficult to understand why they cannot divide by zero, you may find it helpful to use natural numbers in examples and ask students to verify the results. For example, $3 \div 0 = x$ means $3 = 0 \cdot x$. They should see that any replacement for x results in a false statement. Similarly, $0 \div 0 = x$ means $0 = 0 \cdot x$. There are infinitely many replacements for x that result in a true statement.

Provision for individual differences

■ For a class of slow learners, plan to use two periods for this lesson. On the first day, discuss the developmental exercises through the material on the closure property on page 263 and assign "On your own" exercises 1 through 21. Finish the lesson on the second day.

If time permits, use the idea of mapping to provide drill work for slow learners. Supply the student with the image under the mapping and suggest that they designate all the ordered pairs that map onto the image.

If your abler students have a thorough understanding of the ideas presented in the previous lesson, they can complete this lesson as a self-study assignment.

---

6 / 62 Exploring ideas / Pages 265-268

The commutative properties

Objectives

■ The students learn that addition and multiplication of natural numbers have the commutative property. They also learn that subtraction and division do not have the commutative property.

Content overview

■ Since the operation of addition involves the union of disjoint sets, the properties of addition may be derived from such unions. The first prop-

erty that we consider is commutativity. The commutative property of addition is expressed by $a + b = b + a$. For each variable, U = N. To establish commutativity as a property of the addition of natural numbers, we note that, if A and B are any disjoint sets, $A \cup B = B \cup A$. This means that the union of A and B is the same set as the union of B and A. Consequently, the number associated with $A \cup B$ is the same as the number associated with $B \cup A$. We know also that, if A and B are disjoint sets and $a = n(A)$ and $b = n(B)$, then $a + b = n(A \cup B)$. Also, $b + a = n(B \cup A)$. Therefore, $a + b = b + a$ for each replacement of a and b from N.

The commutative property of multiplication is expressed by $ab = ba$. For each variable, U = N. If $a = n(A)$ and $b = n(B)$, then $ab = n(A \times B)$ and $ba = n(B \times A)$. $A \times B$ and $B \times A$ are not the same set (except when A = B), but they are equivalent sets. D7 on page 267 demonstrates that $R \times S$ and $S \times R$ are equivalent. Since the same number is associated with each of two equivalent finite sets, $ab = ba$ for each replacement of a and b from N.

Neither subtraction nor division has the commutative property. $a - b = b - a$ yields a true statement only when the same number is used to replace a and b, and because there are replacements that do not satisfy the condition $a - b = b - a$, subtraction of natural numbers does not have the commutative property. In a similar way, $a \div b = b \div a$ yields a true statement only when a and b are replaced by the same counting number. Because there are replacements that do not satisfy the condition $a \div b = b \div a$, division of natural numbers does not have the commutative property.

We may invent operations and examine them to see what properties they have. We may, for example, define for the set of natural numbers, a binary operation called "mulad" (indicated by the symbol #) along with the usual operations of addition and multiplication. Suppose that mulad involves many-to-one mappings of each member of $N \times N$ onto a unique member of N. The rule is: Find the product of 3 and the first component, and map the ordered pair onto the sum of that product and the second component. An example is: $(2, 4) \rightarrow 2 \# 4$. The standard name of $2 \# 4$ is 10. (10 is obtained from $3(2) + 4$.) This operation is closed in the set of natural numbers.

If the operation of mulad is commutative, then (4, 2) must also map onto 10.

$4 \# 2 = 3(4) + 2$.

$3(4) + 2 = 14$.

$4 \# 2 = 14$.

The operation of mulad is not commutative because (4, 2) and (2, 4) map onto different numbers.

Instruction notes

■ You might introduce this lesson by discussing physical activities in which the result of the activity may or may not be changed by the order in which the activity is carried out. You can, for example, suggest that the students think of the activity of putting on a hat and then a coat. Ask: "Do you obtain the same result when you put on your coat and then your hat?" [Yes] "Think about putting shampoo on your hair and then rinsing your hair with clear water. Do you obtain the same result when you rinse your hair with clear water and then put shampoo on it?" [No] "You know that in some everyday situations you obtain the same result regardless of the order in which you do certain things, and that in other situations you obtain a different result if you change the order. In this lesson you will learn whether or not the same is true in numerical situations."

After exercise J on page 265, introduce sets E and F, such as $E = \{5\}$ and $F = \{10, 15\}$, to your students. Have them use sets E and F to show

hat the sentence $1 + 2 = 2 + 1$ expresses a true tatement.

To increase interest in the commutative prop- rty of addition and the commutative property of nultiplication, you might have the students con- truct a table similar to the one shown below. Give the students time to study the table. Some of he students may discover the symmetry of the rrangement of the numerals with respect to a icture of a diagonal drawn through the table.

×	0	1	2	3	4	5	6	...
0	0	0	0	0	0	0	0	
1	0	1	2	3	4	5	6	
2	0	2	4	6	8	10	12	
3	0	3	6	9	12	15	18	
4	0	4	8	12	16	20	24	
5	0	5	10	15	20	25	30	
6	0	6	12	18	24	30	36	
⋮								

Encourage the students to make the generaliza- tion that the numerals on one side of the picture of the diagonal are symmetrical to the numerals on the other side. This involves the idea of the commutative property of multiplication.

Provision for individual differences

■ Abler students should not experience much dif- ficulty with this lesson. You will find that they readily accept the algebraic generalizations for the properties that are developed in the lesson. Note the "Special challenge" exercises on page 269 that may be used as enrichment with abler students.

Some students may find it interesting to invent operations, some of which are commutative and some of which are not commutative. They may also be intrigued by the task of inventing names for these operations.

For slow learners, stress identification and use of the properties. Do not expect them to display facility with the algebraic generalizations of the properties. The construction of tables such as have been suggested can provide slow learners with valuable drill in computation.

To reduce the teaching time needed for this lesson, you could omit the part of the lesson that deals with the development of the commutative property through set considerations. To do this, omit displays 2 and 3 and developmental exercises B through K on page 265, and omit displays 6 and 7 and developmental exercises N through V on page 266.

6 / Special challenge / Page 269

Objectives

■ The students study an invented operation in the set of the natural numbers. They test for closure, and then they determine whether or not the operation has the commutative property.

Content overview

■ A binary operation on a set of objects involves a pair of objects and is an association of two members of a given set with one member of the set. A binary operation on a set is a special many-to-one mapping of the set of ordered pairs of the objects of a set onto the members of the set. Closure is a fundamental property of number systems. It states that the result of performing an operation on two members of a set must be contained in the given set. Closure is implied in the definition of an operation on a set.

The operation of "star," given in the student's book, maps (2, 5) onto $2 + 2(5)$, or 12. Because (0, 6), (6, 3), and (4, 4) also map onto 12, "star" is a many-to-one mapping.

It can be shown that the operation "star" has closure in N: $2b$ is a natural number because of

the closure property of multiplication, and $a + 2b$ is a natural number because of the closure property of addition.

Of the operations described in this "Special challenge," only that expressed in exercise K has the commutative property.

Observe that the set of natural numbers is not closed under the operations expressed in exercises J and L.

---

6 / 63 Exploring ideas / Pages 269-273

The associative properties

Objectives

■ The students learn that addition and multiplication of natural numbers have the associative property. They also learn that subtraction and division do not have the associative property.

Content overview

■ Remember that addition and multiplication are defined as binary operations. This means that in each operation you work with only two numbers at a time, so that $5 + 6 + 7$ must be interpreted either as $(5 + 6) + 7$ or $5 + (6 + 7)$. The associative property of addition guarantees that $(5 + 6) + 7$ is the same number as $5 + (6 + 7)$. The basis of this property is the associativity of the union of sets; since $(A \cup B) \cup C$ is the same set as $A \cup (B \cup C)$, $(a + b) + c = a + (b + c)$. As a consequence, the numeral $5 + 6 + 7$ names exactly one number. (In lesson 65 the term *expression* is introduced for numerals such as $5 + 6 + 7$. See page 279, *STM 1*.)

Since multiplication is also a binary operation, $5 \cdot 6 \cdot 7$ must be interpreted either as $(5 \cdot 6) \cdot 7$ or as $5 \cdot (6 \cdot 7)$. By virtue of the associative property, $(5 \cdot 6) \cdot 7$ is the same number as $5 \cdot (6 \cdot 7)$. The associative property of multiplication of natural numbers has as its basis the associative property of the Cartesian product of sets; that is, $(A \times B) \times C$ is equivalent to $A \times (B \times C)$.

In general, suppose that a, b, and c are variables for the members of any set and that the symbol * represents an operation; if we say that $a * (b * c) = (a * b) * c$, for each replacement of a, b, and c, then the operation has the associative property.

An important point in the lesson is that, while one example never proves a property, one *counterexample* is sufficient to disprove it. The true statements $(7 - 5) - 0 = 7 - (5 - 0)$ and $6 \div (3 \div 1) = (6 \div 3) \div 1$ suggest that subtraction and division have the associative property. However, a property of an operation must hold for all permissible replacements of the variables in the universe concerned. The statements $(7 - 5) - 1 = 7 - (5 - 1)$ and $(72 \div 12) \div 2 = 72 \div (12 \div 2)$ are false and are called counterexamples. These two counterexamples are enough to prove that subtraction and division do not have the associative property.

The operation of mulad, described in "Content overview" for lesson 62, does not have the associative property. For example, consider whether $(2 \# 3) \# 4 = 2 \# (3 \# 4)$.

$(2 \# 3) \# 4 = 3[3(2) + 3] + 4$.

$3[3(2) + 3] + 4 = 31$.

$2 \# (3 \# 4) = 3(2) + 3(3) + 4$.

$3(2) + 3(3) + 4 = 19$.

Since $(2 \# 3) \# 4 \neq 2 \# (3 \# 4)$, mulad does not have the associative property.

The associative properties can be used to simplify computation. For example, if we are given $59 + 22 + 78$, we can find the standard name for the sum by thinking "$59 + (22 + 78)$" if we know that $22 + 78 = 100$.

Exercises A through G on page 272 provide readiness experience for an idea that is developed in unit 7. This idea is the transitive property of equality; that is, if $a = b$ and $c = b$, then $a = c$.

Instruction notes

Before beginning this lesson, have the students escribe briefly in their own words the closure nd commutative properties. Then say: "You now that the closure and commutative properies are properties of the operations of addition nd multiplication for natural numbers. In this esson you will study another property of each of hese operations."

Emphasize for students that the associative roperty of addition is an order property when a um of *three* addends is to be found. The comnutative property is an order property involving nly two numbers.

After exercise J on page 270, say: "Let us see ow we can use the associative property of addiion to help us in computation." Write on the halkboard "$69 + 3 + 97$." Say: "Suppose that ve want to find what natural number is named by his numeral. Which is the easier way for us to ind the number?" [Find the sum of 3 and 97; hen find the sum of 69 and 100.] "Are you ure that $(69 + 3) + 97 = 69 + (3 + 97)$?" [Yes] 'Why?" [Because of the associative property of ddition]

After exercise C (column 2) on page 271, write n the board "$4 - 3 - 1$." Ask: "Does this symbol name the number 0, or does it name the number 2?" [You cannot decide unless some agreement has been made.] "Suppose that everyone agreed that, in such a case as this, the difference of the first two numbers is to be found; then the difference of the first difference and the third number is to be found. If this agreement were made, would everyone understand that this symbol names 0?" [Yes]

Provision for individual differences

■ Have the abler students show that the associative properties are properties of the natural numbers, and hence they are independent of the system of numeration. Give them examples like $7_{\text{twelve}} + \mathcal{X}_{\text{twelve}} + 4_{\text{twelve}}$ and $3_{\text{five}} \cdot 4_{\text{five}} \cdot 2_{\text{five}}$. Let them group each numeral in two ways and simplify each numeral.

For slow learners, again emphasize the applications of the properties and minimize the discussion of the conditions $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.

For slow learners, you could devote two days to this lesson. You could discuss the developmental exercises and assign "On your own" exercises 1 through 10 on the first day. On the second day, discuss the commutative and associative properties again and assign "On your own" exercises 11 through 34.

Give slow learners help in getting started with "On your own" exercises 11 through 34, and then give them as much individual help as possible as they work on these.

6 / 64 Exploring ideas / Pages 274-277

The distributive property

Objectives

■ The students learn what it means for an operation to distribute over another operation. They learn that multiplication distributes over addition.

Content overview

■ We have defined two binary operations, addition and multiplication, under which the set of natural numbers is closed. Each of these operations is commutative and associative, and they are linked together by means of the distributive property.

The distributive property of multiplication over addition may be illustrated by grouping a set of objects in the way that the 20 objects on

page 274 of *STM 1* are grouped. The first grouping suggests the numeral $(4 \cdot 2) + (4 \cdot 3)$, and the second grouping suggests the numeral $4(2 + 3)$. The true statement $(4 \cdot 2) + (4 \cdot 3) = 4(2 + 3)$ suggests that multiplication distributes over addition. Multiplication and addition of natural numbers are always related in this way. For each replacement of a, b, and c in the universe N, a true statement is always obtained from $ab + ac = a(b + c)$.

The distributive property of multiplication over addition may be proved by using set ideas. Suppose that $a = n(A)$, $b = n(B)$, $c = n(C)$, and sets B and C are disjoint. Then, by using the definitions of sum and product:

$b + c = n(B \cup C)$.

$a(b + c) = n[A \times (B \cup C)]$.

$ab = n(A \times B)$.

$ac = n(A \times C)$.

$ab + ac = n[(A \times B) \cup (A \times C)]$.

By using the definition of union and the Cartesian product of sets:

$A \times (B \cup C) = (A \times B) \cup (A \times C)$.

$n[A \times (B \cup C)] = n[(A \times B) \cup (A \times C)]$.

Therefore, $a(b + c) = (ab) + (ac)$.

Mathematicians speak of a *left-hand* and a *right-hand* distributive property. They regard the condition $a(b + c) = ab + ac$ as distinct from the condition $(b + c)a = ba + ca$. For numbers, each of these conditions follows from the other by applying the commutative property of addition.

In lesson 64, the "left-hand" distributive property is developed by using examples, and the property is accepted without proof. Exercises O through U on page 275 develop a proof of the "right-hand" distributive property. This proof involves the "left-hand" distributive property and the commutative property of multiplication. The logic and the mechanics in this proof are important. More proofs are developed in unit 7. Throughout the *STM* program, work with proofs is gradually extended.

The first block of exercises on page 276 o *STM 1* presumes a previous use of the distribu tive property in multiplication computation. Fo example, the distributive property is used to fin the product of a number expressed by a two-digi numeral and a number expressed by a one-digi numeral.

Multiplication distributes over subtraction i we consider as a universe a subset of the natura numbers, that is, when $b - c$ is a natural number Note that $b - c$ is a natural number only when b is replaced by a number that is equal to or greate than the replacement for c. Using this particula universe, a true statement is always obtained from $a(b - c) = ab - ac$.

The knowledge that multiplication sometimes distributes over subtraction can be helpful in performing computation. Suppose that the standard name of the product $75 \cdot 99$ is to be found. Since $99 = 100 - 1$, $75 \cdot 99 = 75(100 - 1)$. Therefore, $75 \cdot 99 = 7500 - 75$.

The distributive property is also used in algebraic factoring. For example, $2x^3 - 14x^2 + 16x$ may be thought of as $2x(x^2 - 7x + 8)$.

Instruction notes

■ Blocks or a flannel board with disks may be used to aid in the development of exercises A through G on page 274.

In discussing exercise A on page 274, make sure students recall that the raised dot indicates multiplication and that parentheses are helpful in thinking about grouping numbers for multiplication as well as to ensure that the addition is not done before the multiplication.

After exercise K on page 275, ask: "What is the common multiplier expressed in each sentence in D2?" [2 in sentence D, 8 in E, and 17 in F]

Before exercise M on page 275, write "(ab)" on the chalkboard. Ask: "What does this symbol indicate?" [A product formed for each replacement of a and b]

After exercise K on page 277, say: "Let us see we can use this idea to help us with computaon. Suppose that we want to find the product of 5 and 98. How can we use the idea presented in xercise K to help us find the product of 25 and 3?" Let the students think about it. [$25 \cdot 98 = 5(100 - 2)$. $25(100 - 2) = 2500 - 50$, or 2450.] ive them a few more examples to work out ientally, such as the following: $30 \cdot 59$, $87 \cdot 99$, nd $15 \cdot 190$.

Provision for individual differences

A class of abler students should be able to comlete this lesson in one day. For average students nd slow learners, you may find that it is necesary to devote two days to this lesson. On the rst day, complete the developmental exercises rough the first paragraph on page 276 (or exerise N on page 275 for slow learners), and assign On your own" exercises 1 through 10, 12 rough 15, and 17, 18, and 19. On the second ay, finish the lesson and assign "On your own" xercise 16 and exercises 20 through 36. For exerises 28 through 36, it may be helpful to have a lass of slow learners determine the common facors in a general class discussion before the stuents begin their assigned work.

Have the abler students test examples such as seven $(4_{\text{seven}} + 6_{\text{seven}}) = (5_{\text{seven}} \cdot 4_{\text{seven}}) + (5_{\text{seven}} \cdot$ seven) to see if the distributive property holds for nother system of numeration. [The property olds because it is a property of number systems, ot of numeration systems.]

If your class is not able to maintain the time chedule you have planned, or if you have a class f slow learners, you can omit the part of the leson dealing with the proof of the right distributive property, that is, displays 4 and 5 and developmental exercises O through U on page 275. n place of these exercises, use examples similar o those in D1, D2, and D3 to provide students vith practice in applying the property.

6 / 65 Exploring ideas / Pages 279-281

Simplifying expressions

Objectives

■ The students learn what is meant by expressions and by standard names of numbers. They learn how to simplify an expression.

Content overview

■ Numbers are expressed in either of two ways, by standard names or by expressions. A standard name, for example, 36, is usually simpler than the expression $4(12 - 3)$, even though both symbols name the same number. Thus, it is often desirable to simplify an expression to find the standard name of the number. Any numeral that contains one or more symbols for operations is an expression. The basic ideas that are involved in simplifying expressions are developed in this lesson.

Notice that the symbols $+$, $\times$, $\div$, and $-$ are referred to in the lesson as operation symbols. Of course, subtraction and division of natural numbers are not considered to be operations; they are described in terms of their relation to the operations of addition and multiplication. However, to simplify an expression, we think about computation rather than about the mathematical meaning of an operation. Computation involves the familiar four processes. Because simplifying expressions involves computation, the symbols $\div$ and $-$ are referred to as operation symbols.

A product may be indicated in three different ways: by the symbol $\times$, by a raised dot, and by parentheses. Hence, 4(6) is a product. Parentheses are also used to indicate the order in which computation is to be performed. For example, to simplify the expression $(4 + 8) \div 2$, the standard

name of $4 + 8$ should be found first; then 12 is divided by 2. To simplify $4 + (8 \div 2)$, the standard name of $8 \div 2$ should be found first; then 4 is added to 4.

In expressions that do not contain parentheses and in which there is at least one multiplication symbol ($\times$ or $\cdot$) or division symbol ($\div$) and at least one addition symbol ($+$) or subtraction symbol ($-$), there is a rule to indicate the order in which the computation is to be performed. The multiplication or division should be done first; then the addition or subtraction; that is, $5 + 3 \cdot 6$ is interpreted as $5 + (3 \cdot 6)$. Also, $20 \div 4 + 6$ is interpreted as $(20 \div 4) + 6$.

In those expressions that contain only addition or subtraction symbols, or only multiplication or division symbols, the computation is performed from left to right. For example, $30 \times 15 \div 3$ is interpreted as $(30 \times 15) \div 3$, and $14 - 6 + 4$ is interpreted as $(14 - 6) + 4$.

These rules do not have a mathematical basis. They are conventions that have been agreed upon so that people will understand one another.

Instruction notes

■ After exercise I on page 280, write the symbol 4(7) on the chalkboard. Say: "Is this symbol an expression?" [Yes] "What operation do you use to find the standard name of the product 4 times 7?" [Multiplication] "Is the parentheses in this expression used as an operation symbol?" [Yes]

Before exercise N on page 280, say: "In the expressions you have worked with so far, how was the order in which you were to compute indicated?" [By parentheses] "Now you will learn some rules you can use to simplify expressions that do not contain parentheses."

After exercise W on page 281, write the expression $5(2 + 3)$ on the chalkboard. Say: "Does this expression name the same number as 5 times 2 plus 3?" [No. $5 \cdot 2 + 3$ is an expression for 13. $5(2 + 3)$ is an expression for 25.] "How would you read the symbol $5(2 + 3)$?" [5 times the sum 2 plus 3]

Provision for individual differences

■ A class of abler students could be given this lesson as a self-study assignment. Abler students might also be asked to simplify the expressions included in the sentences in "On your own" exercises 27 through 35.

The last nine "On your own" exercises might be very difficult for slow learners. Consider omitting these exercises from the assignment for these students, or complete them as "board work" and give students as much help as possible.

---

6 / 66 Exploring ideas / Pages 282-284

The natural-number system

Objectives

■ The students learn that the natural-number system includes a set, two operations, and some properties of these operations.

Content overview

■ In this lesson many of the ideas that have been developed so far in unit 6 are synthesized into the idea of the natural-number system. The idea of a system is important because it leads to an understanding of what is meant by a structure in mathematics.

The natural-number system is the first of several number systems that are developed in the *STM* program. The even-number system is also developed in this lesson. A finite number system is developed in lesson 67; the system of the rational numbers of arithmetic is developed in unit 7; the system of the rational numbers (including the positive rationals, the negative rationals, and zero), along with certain subsets of

ıe set of rationals that are number systems, is eveloped in *STM 2*; and the real-number system also considered in *STM 2*.

Each of the characteristics of a number sys-:m has been carefully developed in lessons 58 ırough 64. Following is a list of these character-tics:

1 There is a set of objects.
2 Two operations are defined for the set.
3 Each operation has a closure property.
4 Each operation is commutative.
5 Each operation is associative.
6 One of the operations distributes over the ther.

There is a slight variation among the defini-ons of a number system as they appear in mathe-ıatical literature, but all definitions include a set f objects and two operations. The definition de-eloped in *STM 1* is the easiest for the student o understand.

To determine if a set (together with the opera-ons and certain of their properties) forms a umber system, it is necessary to check each re-uirement systematically. Consider set S and he operations of union and intersection when $= \{\{a, b\}, \{a\}, \{b\}, \{\ \}\}$. Note that we are ow considering two operations that differ from he usual operations of addition and multiplica-ion. Note also that union and intersection are perations on sets and each member of S is a set. Ve can consider:

1 A set of objects
2 Two operations, that is, union and intersection
3 That the set is closed under each operation, $\cup y$ is a member of S, and $x \cap y$ is a member f S when the universe for x and y is S
4 That each operation is commutative, that is, $\cup y = y \cup x$, and $x \cap y = y \cap x$
5 That each operation is associative, that is, $\cup (y \cup z) = (x \cup y) \cup z$, and $x \cap (y \cap z) = x \cap y) \cap z$
6 That intersection distributes over union, that is, $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$

Therefore, set S, together with the operations of union and intersection and some of their prop-ties, forms a number system.

It can be shown that the set of even numbers, together with two operations and some of their properties, is a number system. To show that the set is closed under addition, the following can be used: If $2x$ is one member of E and $2y$ is another member, then their sum is $2x + 2y$. The universe for x and y is N. By the distributive property, $2x + 2y = 2(x + y)$. After x and y are replaced, $2(x + y)$ has a factor of 2, and the sum $2x + 2y$ is a member of E.

The product of two members of E is $2x(2y)$. The universe is N. Because $2x(2y) = 2(x \cdot 2y)$, and $2(x \cdot 2y)$ has a factor of 2, the product $2x(2y)$ is a member of E.

Each operation also has a commutative prop-erty and an associative property, and multiplica-tion distributes over addition.

A good way to begin testing a subset of the natural numbers, together with the operations of addition and multiplication, to find whether or not it forms a number system is to see if the set is closed under each operation. For a finite set, find the sum or product that is formed by the two greatest members. For an infinite set, test several pairs of members. Remember that if you find one instance in which the set is not closed, it means that the set, with its operations and their proper-ties, is not a number system.

Instruction notes

■ You might begin the lesson in the following way. Say: "Early in this unit you learned to think of the natural numbers as properties of sets." Ask: "How do we tabulate the set of natural numbers?" Write on the chalkboard "{0, 1, 2, . . .}." Ask: "What two operations of

the natural numbers have you studied?" Write the headings "Addition" and "Multiplication" under the tabulation. Ask: "Is the set of natural numbers closed under both operations?" Write "$a + b$ is a member of N" under "Addition." Write "ab is a member of N" under "Multiplication." Ask: "Does each operation have a commutative property?" In the column for addition write "$a + b = b + a$." In the column for multiplication write "$ab = ba$." "What other property does each operation have?" [Associative] Write under the proper headings "$(a + b) + c = a + (b + c)$" and "$(ab)c = a(bc)$." Ask: "What property involves the two operations?" [Distributive] Write "$a(b + c) = ab + ac$." Say: "The set of natural numbers, together with these two operations and their properties, forms a number system."

Explain to the students that the study of number systems is important because it will help them understand that mathematics is a study of structures. Also tell them that the system of natural numbers is only the first of many systems they will work with as they continue their study of mathematics.

Have your students turn to page 282 and briefly review the characteristics of the natural-number system as described in D1. Then continue with the paragraph that follows exercise E on page 282.

In studying exercises L and M on page 283, make certain your students understand that E is closed under both operations because both $2(m + n)$ and $2(m \cdot 2n)$ have a factor of 2. Any natural number that has a factor of 2 is an even number.

Provision for individual differences

■ Have your abler students show that {0, 10, 20, . . .}, together with addition and multiplication, forms a number system, when U = N. [There are a set and two operations. Because $10m + 10n = 10(m + n)$, the set is closed under addition. Because $10m(10n) = 10(m \cdot 10n)$, the se is closed under multiplication. Each operation i commutative and associative, and multiplicatio distributes over addition.]

For a class of slow learners, develop most o the ideas in this lesson at the chalkboard. Intro duce these ideas by using numerals instead o placeholders for the properties. For example write "$5 + 3 = 3 + 5$" rather than "$a + b = b + a$" for the commutative property of addition. Whe it appears that slow learners understand the spe cific cases involving numerals, it may be worth while to expose them to the general cases using placeholders.

Remember that the idea of a system of num bers is a sophisticated mathematical idea and tha only your abler students may understand the sig nificance of this idea at this stage of their de velopment.

---

6 / 67 Exploring ideas / Pages 285-288

A finite number system

Objectives

■ The students learn how to decide whether o not a finite set of natural numbers, together with two operations, forms a number system.

Content overview

■ This lesson introduces number systems in which the set of numbers is finite. Such a number system is often called a modular system. A modular system, then, is composed of a finite set of numbers, two operations, three properties of each operation, and a distributive property that relates the operations.

The idea of finite number systems has applications in everyday living, as, for example, the set of days in a week. We can find, by counting, that

ne day beyond Wednesday is Thursday, that five ays beyond Wednesday is Monday, and so on. his information may also be obtained by using hat is called modular arithmetic and, in this articular case, a modulus-seven (mod-7) system. Ve can establish a one-to-one correspondence etween the set of days in a week, {Sunday, Monday, . . ., Saturday} and {0, 1, 2, . . ., 6}. he problem of finding one day beyond Wednesay is related to the modular arithmetic problem + 1. The answer is 4(mod 7), or Thursday, in ie correspondence established previously. Simiarly, five days beyond Wednesday is 3 + 5, or (mod 7).

Modular arithmetic is sometimes called "reainder arithmetic." Note, for example, that if e want to find ten days beyond Wednesday by sing mod-7 arithmetic, we can divide 10 by 7 to btain the quotient 1 and a remainder 3. The reainder is "added to" Wednesday, 3 + 3, and we btain the answer 6(mod 7), which corresponds o Saturday.

The set of four seasons may be associated with mod-4 system. The use of the 12-hour clock ay be associated with a mod-12 system ("clock rithmetic").

Consider the set of natural numbers mod 6. he set of objects is {0, 1, 2, 3, 4, 5}. The sysem includes two operations that we call addiision and muldivision. To find the addiv of any wo members of the set, just find the sum and hen divide the sum by 6. The remainder is the ddiv of the two numbers. For example, suppose hat the addiv of 3 and 5 is to be found. The sum f 3 and 5 is 8. 8 divided by 6 is 1, remainder 2. herefore, $3 \oplus 5 = 2$. (Here the symbol $\oplus$ repreents the operation of addivision.)

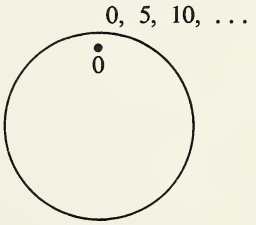
To find the muldiv of two objects, first find the roduct, and then divide the product by 6. The emainder is the muldiv of the two objects. Supose that the muldiv of 5 and 5 is to be found. The product of 5 and 5 is 25. 25 divided by 6 is 4, remainder 1. Therefore, $5 \otimes 5 = 1$.

For the set of natural numbers mod 6, there are two operations, and the set is closed under each operation. Because each operation is commutative and associative, and muldivision distributes over addivision, mod 6 is a number system.

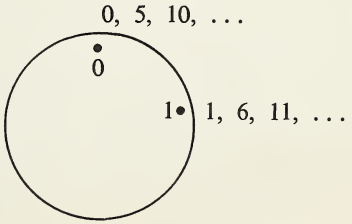
In a finite number system, it is possible (though it may become tedious) to test all combinations of members to see if the set is closed under each operation and to find if each operation has the commutative and the associative properties.

Instruction notes

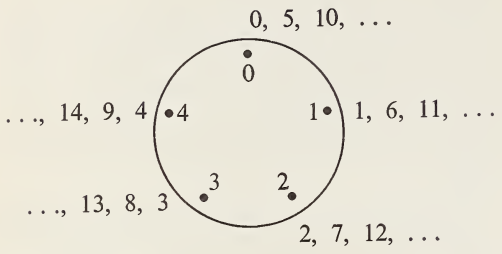
■ After exercise E on page 285, say: "Think of the natural numbers that you can divide by 5 with a remainder of 0. Is 0 such a number?" [Yes] "What are some other numbers?" [5, 10, 15, . . .] Make this sketch on the chalkboard:



Say: "Think of the natural numbers you can divide by 5 with a remainder of 1. What are some of these numbers?" [1, 6, 11, . . .] Include the numerals in your sketch, as follows:

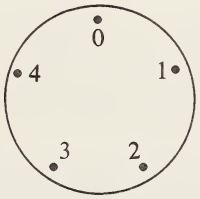


Follow the same line of questioning, and include these numerals:



Show that the remainders obtained from dividing each of the first fifteen natural numbers by 5 are indicated in the picture. Ask: "What is the remainder when 15 is divided by 5?" [0] "Does 15 belong with these numbers?" (Indicate the sequence 0, 5, 10, . . .) [Yes] "Where do 16 and 17 belong?" [With the remainder 1 and remainder 2 sequences, respectively]

After discussing exercise D (column 2) on page 285, make another sketch that looks like the following:



Say: "There is another way to find the addiv of two members of set T. Look again at D3. Sentence A expresses the idea that the addiv of 4 and 3 is 2. Look at the sketch on the board. I am going to begin at zero and then count four spaces, going clockwise. Where do I stop?" [At point 4] "Now I am going to count three more spaces. Where do I stop?" [At point 2] "Is 2 the addiv of 4 and 3?" [Yes] Follow the same procedure for sentences B, C, and D in D3.

If you used an addition table and a multiplication table in lesson 62 to develop the commutative properties of the operations on natural numbers, you could use the tables in D4 and D6 on page 286 to develop the commutative properties of the operations on the members of set T. If you do this, you could omit exercises A through F on pages 286 and 287.

Provision for individual differences

■ Have abler students work the exercises below.

These exercises refer to set T and the operations studied in the lesson. Tabulate the solution set of each condition. U = T.

1 $4 \oplus 4 = x$. [{3}]
2 $3 \otimes 3 = y$. [{4}]
3 $x \oplus 2 = 4$. [{2}]
4 $4 \otimes y = 2$. [{3}]
5 $(3 \oplus x) \oplus 2 = 0$. [{0}]
6 $0 \otimes (4 \otimes x) = 2$. [{ }]
7 $2 \otimes (3 \oplus x) = (2 \otimes 3) \oplus (2 \otimes 2)$. [{2}]

For a class of slow learners, you may find it helpful to use a model of a clock for teaching modular arithmetic. Cut a disk out of tagboard, write the numerals 0, 1, 2, 3, and 4 (for a mod-5 system) on the disk, and use a paper fastener to attach a "hand" to the disk. The students can use the clock to demonstrate "adding."

We suggest that you do not assign "On your own" exercises 10 through 19 to your slow learners.

If your class is not able to maintain the time schedule you have planned, or if you have a class of slow learners, you can omit this lesson.

6 / 68 Exploring ideas / Pages 289-292

Zero and one

Objectives

■ The students learn that addition and multiplication have certain special properties that concern zero and one.

Content overview

In this lesson we develop the special properties r the two operations of the natural-number sys- m that concern the numbers zero and one. We so explain how these properties relate to sub- action and division involving natural numbers.

In the natural-number system, 0 is the identity ement for addition. The identity-element prop- ty of addition arises from study of the condition $+ x = x$. Suppose that $x = n(B)$ and $0 = n\{\ \}$. he sum $0 + x$ is the number of members in $\} \cup B$ because the union of a given set and the npty set is the given set. Thus, $0 + x$ and x are ie same number, and $0 + x = x$ is a property of ie natural-number system.

Consider an object e, a member of set S, with peration # defined on S. In general, if a true atement is obtained for each replacement of x y a member of S in $e \# x = x$, then e is the lentity element for operation #.

Using the identity-element property of addi- on and the commutative property of addition of atural numbers, we can show that $x + 0 = x$ is lso a property of the natural-number system. he relation of subtraction to addition and the lentity-element property of addition of natural umbers also lead to $x - x = 0$ and $x - 0 = x$ as roperties of the system of natural numbers.

The zero property of multiplication is the ondition $0 \cdot x = 0$. Suppose that $x = n(K)$ and $= n\{\ \}$. The product $0 \cdot x$ is the number of iembers in $\{\ \} \times K$. Because there are no mem- ers in the empty set to pair with the members of , $\{\ \} \times K$ does not contain any ordered pairs. herefore, $\{\ \} \times K = \{\ \}$. Since the number of iembers in $\{\ \} \times K$ is 0, $0 \cdot x = 0$. Using this roperty and the commutative property of multi- lication, $x \cdot 0 = 0$ is obtained as a property.

The condition $x/0 = y$ has no solution. $x/0 = y$ ieans $x = 0 \cdot y$. There is no counting number iat, when multiplied by 0, gives a number other than zero. Therefore, symbols such as 1/0, 2/0, and 3/0 do not name numbers. Because every member of N satisfies $0/0 = x$, we say that 0/0 is indeterminate. Consequently, we say that division by zero is impossible because $0 \div 0$ is indeterminate and $1 \div 0$, $2 \div 0$, and so on, are meaningless symbols.

In the natural-number system, 1 is the identity element for multiplication. The identity-element property of multiplication is the condition $1 \cdot x = x$. Suppose that $1 = n\{0\}$ and $x = n(E)$. The Cartesian set $\{0\} \times E$ is the set of ordered pairs generated by matching 0 with each member of set E. Because $\{0\} \times E$ and E are equivalent sets, the same number is associated with each set. Therefore, $1 \cdot x$ and x are the same number. By using the commutative property of multiplication, the property $x \cdot 1 = x$ is obtained from $1 \cdot x = x$.

By using the relation of division to multiplication and the identity-element property, the property $x/1 = x$ is obtained. By using the property $x \cdot 1 = x$ (or $x = x \cdot 1$), the property $x/x = 1$ is obtained. The universe for x is C.

A number system may not have the two identity-element properties. A modular-6 system has 0 and 1 as identity elements. The counting-number system does not have an identity element for addition, but it does have an identity element for multiplication. The even-number system has an identity element for addition, but it does not have an identity element for multiplication.

Notice that a diagonal bar is introduced on page 291 to indicate division. Students learn from the context whether the symbol 12/4 names a rate pair or a quotient. The symbol for equivalence may be used with a symbol that names a rate pair (12/4), but the symbol for equivalence may not be used when that symbol (12/4) expresses a quotient.

Instruction notes

■ After discussing exercise H on page 289, write on the chalkboard:

$$0 + 0 = 0.$$
$$0 + 1 = 1.$$
$$0 + 2 = 2.$$
$$0 + 3 = 3.$$
$$\vdots$$

Say: "The dots tell you that the list of sentences is endless. Does each of the sentences express a true statement?" [Yes] "Is each of the statements true because of the identity-element property of addition?" [Yes]

Before exercise J on page 291, write on the board:

$$1 \cdot 0 = 0.$$
$$1 \cdot 1 = 1.$$
$$1 \cdot 2 = 2.$$
$$1 \cdot 3 = 3.$$
$$\vdots$$

Ask: "Does each of these sentences express a true statement?" [Yes] "Each statement is true because of what property?" [Identity-element property of multiplication of natural numbers]

After exercise N on page 291, say: "Think about the set of counting numbers. Does this set contain an identity element for addition?" [No] "Does it contain an identity element for multiplication?" [Yes] "A set may contain identity elements for two operations, for one operation, or for neither operation that is defined on the set."

Provision for individual differences

■ Have your abler students name the identity element for each operation defined on the members of set F expressed in D8 on page 288. [0 for addivision, 1 for muldivision] Also, have them name the identity elements for addition and multiplication for each set named in the next column on this page of the *Teaching Guide*.

{0, 5, 10, . . .} [0 for addition, none for multiplication]

{1, 10, 100, . . .} [none for addition, 1 for multiplication]

With your slow learners, emphasize the use of each property. Use examples such as $8/8 = 15/15 = 1$, $125/125 = 1$, and so on, to help develop the condition $a/a = 1$.

If your class is not able to maintain the time schedule you have planned, or if you have a class of slow learners, you can omit the part of the lesson dealing with identity elements as they are developed using set ideas. To do this, omit display and developmental exercises A through G on page 289, and displays 4 and 5 and developmental exercises A through H on page 290.

---

6 / Special challenge / Page 293

Objectives

■ The students relate subdivision to the operation addivision, relate didivision to the operation multidivision, and find solutions of conditions that involve a finite number system and its operations.

Content overview

■ Addivision and muldivision tables can be used to find the solutions of the conditions expressed in exercises C through J. For example, the solution of $3 \oplus 4 = x$ is the number named in the column under the numeral 3 and opposite the numeral 4 in the addivision table.

The tables can also be used to find the solutions of conditions that involve subdivision and didivision. For example, consider $3 \ominus 1 = x$. $3 \ominus 1 = x$ also means $3 = 1 \oplus x$. The solution of $3 = 1 \oplus x$ is the second component of the ordered pair whose first component is 1 and which maps onto 3 by addivision. We can look under the column headed by the numeral 1 to find the addi

. Since the addiv 3 is opposite 2, we know that ıe solution of $3 = 1 \oplus x$ is 2 and the solution of $\ominus 1 = x$ is 2.

Observe that addivision and muldivision each ave an identity element and that set T is closed nder subdivision and didivision.

/ 69 Exploring ideas / Pages 295-298

Composite numbers and prime numbers

Objectives

The students study certain ideas of elementary umber theory. Specifically, these ideas are prime umbers, composite numbers, and twin prime umbers.

Content overview

A first number is divisible by a second number f, when the first number is divided by the second umber, the remainder is 0. A prime number is a ounting number that is divisible by exactly two ounting numbers. This means that the natural umber 1 is *not* a prime number. To define prime umbers so that 1 is not prime is a practice that is ommonly followed in number theory.

A composite number is a counting number that s divisible by at least three counting numbers.

If we can arrange a set of objects in a rectangular pattern of at least two rows, with at least two bjects in each row, then the number associated vith the number of objects is composite. The least composite number is 4. Four objects can be arranged in this way:

★ ★
★ ★

The next composite number is 6. Six objects can be arranged in this way:

★ ★ ★
★ ★ ★

In contrast, if a number is a prime number, then we cannot associate it with a set of objects that can be arranged in a rectangular pattern so that there are at least two rows and at least two objects in each row.

The number 2 is unique in that it is the only even prime number. An even number is a natural number that has 2 as a factor. An odd number is any counting number that does not have 2 as a factor. An odd number can also be described as the sum of 1 and an even number.

The Sieve of Eratosthenes is a chart that provides a systematic way to find all primes between 1 and a given number. To use this method, first list the numerals for the numbers from 1 through 100 (or any other number). Beginning with 4, cross out the name of each number that has a factor of 2. Next, start with 6 and cross out the numeral for each number that has a factor of 3. Then cross out the numeral for each number that has a factor of 5, of 7, and so on. When this procedure of crossing out numerals is continued, a list of all the prime numbers that are less than 100 is obtained. Notice that, when a numeral (such as 14) has been crossed out as the name for a multiple of a lesser prime (such as 2), it need not be crossed out again if it also names a multiple of a greater prime (such as 7).

Pairs of numbers such as 3 and 5, 5 and 7, 29 and 31, and 71 and 73 are twin primes. Twin primes are pairs of prime numbers that have exactly one counting number between them. For example, 60 is the only counting number between the twin primes 59 and 61.

The prime numbers 3, 5, and 7 represent the only instance of triple primes. For any other group of three consecutive odd numbers, one of the numbers is divisible by 3.

The set of prime numbers is an infinite set. If the set of primes were a finite set, there would be

a greatest prime number. The following argument proves that there is no greatest prime number.

Suppose that the set of primes is {2, 3, 5, . . ., x} and that x is the greatest member of this set. Then the product $2 \times 3 \times 5 \times \ldots \times x$ is greater than x and is divisible by each member of the set of primes. Obviously, the sum of this product and 1 is not divisible by any one of the members of the set {2, 3, 5, . . ., x}. Consequently, that sum is either a prime number itself or it is divisible by some prime number not in the set {2, 3, 5, . . ., x}. In either event, there must be prime numbers greater than x. Therefore, x is not the greatest prime number. Since the set of prime numbers does not have a greatest member, it is an infinite set.

Three more ideas of number theory are developed in lesson 70. In *STM 2* the ideas of elementary number theory that are developed include the number of divisors of a given number and the sum of the divisors of a given number.

Instruction notes

■ If possible, duplicate charts of the numerals from 1 through 100 so that each student may make a Sieve of Eratosthenes. (See D2, D3, and D4 on pages 296 and 297.)

Ask a few students to make a large chart of the Sieve to put on the bulletin board. This could be a project for the abler students, to be completed before the lesson is taught to the class, or it could be a project for slow learners, to be completed after the lesson has been presented.

After exercise Q on page 296 has been discussed, you could have the students finish the lesson by themselves. When they have finished the developmental exercises, review the following ideas with them:

1 A prime number is a counting number that has exactly two factors.

2 A composite number is a counting number that has at least three factors.

3 Two is the only even prime.

4 A number that is the square of a prime has ex
actly three factors.

5 Primes do not occur at regular intervals in th
ordered sequence of counting numbers.

6 Twin primes are pairs of prime numbers whos
difference is 2.

7 It appears that any even composite numbe
can be expressed as the sum of two primes, bu
no one has found a way to prove this.

8 There is at least one prime number betwee
any number greater than 1 and two times th
number.

Provision for individual differences

■ For a class of abler students, the entire lesso
might be assigned for independent study.

You could also have abler students do som
research on such famous mathematicians a
Gauss, Fermat, Goldbach, or Wilson and mak
informal reports to the class on the contribution
of these men.

Give slow learners as much help as needed
when they are making their charts of prime num
bers. You may find it necessary to concentrate o
the identification of prime numbers and compos
ite numbers and to spend less time on items
through 8 of the review items above. Plan to us
a few minutes of class time each day for a revie
of important ideas previously discussed.

You might want to explain to the students tha
the ideas developed in this lesson are ideas o
elementary number theory. Have them make
list similar to that in the "Instruction" section o
these lesson notes.

If your class is not able to maintain the tim
schedule you have planned, or if you have a clas
of slow learners, you can omit the part of the les
son dealing with the notion of twin primes, th
part that deals with even numbers as the sum o
two prime numbers, and the part about the num
ber of primes between a given number and twic

e given number. To do this, omit displays 5 nd 6 and developmental exercises A through I n pages 297 and 298.

/ 70 Exploring ideas / Pages 299-302

he unique factorization property

Objectives

The students study three more elementary umber-theory ideas: the complete factorization f natural numbers greater than 1, the unique ctorization property, and the greatest common ctor of two or more counting numbers.

Content overview

The *complete factorization* of a number is a pecial way of thinking about the number. The ven number is thought of as the product of all rime numbers for which the given number is the roduct. A prime number can be used more than nce in the complete factorization of the number. /hen the complete factorization of 18 is thought f as $3^2 \times 2$, 3^2 must really be thought of as $\times 3$ rather than as 9, since 9 is not a prime umber.

The complete factorization of a prime number, or example, 13, is simply the given prime number, not $13 \cdot 1$, because 1 is not prime.

Because the numbers 0 and 1 are neither prime umbers nor composite numbers, no complete actorization is defined for these numbers.

Observe the distinction between the complete actorization of a number and the set of factors f that number. The set of factors of 24 is 1, 2, 3, 4, 6, 8, 12, 24}. This set includes very natural number that is a factor of 24, including 24, because every number is a factor of itself. The set of factors of a number always contains 1 and the number itself, as well as all prime

and composite numbers that are factors of the number. Any given factor appears only once in a set of factors. This is not true of the complete factorization of a number. In the complete factorization of a prime number or a composite number, only prime numbers are used.

Every counting number greater than 1 has exactly one complete factorization, since the ordering of the prime factors does not matter. Thus, 2×3 is regarded as the same factorization as 3×2. The statement expressed (by the first sentence in this paragraph) is known as the unique factorization property of the counting numbers.

There are ways of finding the complete factorization of a number when the factorization is not obvious. For example, given the number 168, we may think as follows:

$168 = 2 \cdot 84.$
$2 \cdot 84 = 2 \cdot 2 \cdot 42.$
$2 \cdot 2 \cdot 42 = 2 \cdot 2 \cdot 2 \cdot 21.$
$2 \cdot 2 \cdot 2 \cdot 21 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7.$
$2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 = 2^3 \cdot 3 \cdot 7.$

We may also think:

$168 = 8 \cdot 21.$
$8 \cdot 21 = 2^3 \cdot 3 \cdot 7.$

One technique for finding the complete factorization of a number is to list the prime factors used as divisors, beginning with the least factor, and dividing to obtain a new quotient. Each "new" quotient is divided, and each factor is used exhaustively until the resulting quotient is a prime number. The complete factorization is obtained by listing the divisors used.

Quotients	Divisors
420	2
210	2
105	3
35	5
7	7
1	

$2^2 \times 3 \times 5 \times 7$ is the complete factorization of 420.

The *greatest common factor* of two or more counting numbers is the greatest number that is a factor of the two or more numbers under consideration. The greatest common factor may be found by using either the set of factors of each number or the complete factorization of each number.

A way of finding the greatest common factor of 16 and 52 is to find the intersection of the two sets of factors. The set of factors of 16 is {1, 2, 4, 8, 16}. The set of factors of 52 is {1, 2, 4, 13, 26, 52}. The intersection of these two sets is {1, 2, 4}. The greatest member of the intersection, 4, is the greatest common factor of 16 and 52.

The ability to find the greatest common factor is useful in applying the reduction property of rational numbers. This property is introduced in lesson 72 on page 317.

The complete factorization of counting numbers can be used in obtaining the *least common multiple* of two or more counting numbers. The idea of the least common multiple is used in performing addition and subtraction computations with the rational numbers of arithmetic, as presented in lesson 77. Readiness for the idea of the least common multiple is developed in "On your own" exercises 27 through 32 of lesson 70.

In a mathematics course, some ideas are developed in preparation for the introduction of more complex ideas. This is particularly true of some of the ideas developed in this lesson. Some students may not fully grasp the significance of a greatest common factor or a least common multiple until they have used these ideas in working with the set of rational numbers of arithmetic.

Instruction notes

■ Begin the lesson by saying: "In this lesson you are going to learn more about ideas you studied in the last lesson. Is 20 the product of 4 and 5? [Yes] Write on the chalkboard "$20 = 4 \times 5$." Ask: "4 is the product of what primes?" [2 and 2] Write on the board "$4 \times 5 = 2 \times 2 \times 5$." Say: "When we say that 20 equals 2 times 2 times 5, we are thinking of 20 as the product of prime numbers." You may choose to use only class discussion and the displays on page 299 to develop the idea of complete factorization and then begin the developmental exercises with the first sentence on page 300.

After exercise S on page 301, write the numeral 1308 on the chalkboard. Say: "There are systematic ways to find the complete factorization of a number. What rule can you use to find if a number has a factor of 2?" [A student may suggest that, if the last digit in a numeral expresses an even number, the number has a factor of 2.] Ask: "Is 1308 divisible by 2?" [Yes]

"Think of the sum of the numbers expressed by the digits in the numeral 1308. What is the sum of 1 plus 3 plus 0 plus 8?" [12] "Because this sum is divisible by 3, 1308 is also divisible by 3.

"You can find if a number is divisible by 4 by just considering the number expressed by the digits in the tens' and the ones' positions. Is 130 divisible by 4?" [Yes. 8 is divisible by 4.] "Is 132 divisible by 4?" [No. Twenty-two is not divisible by 4.]

"You know that, if the last digit in a numeral is either 5 or 0, then the number expressed by the numeral is divisible by 5. If you know that a number is divisible by both 2 and 3, do you also know that the number is divisible by 6?" [Yes] "You might experiment to find more rules that you can use to test for divisibility."

Provision for individual differences

■ A class of abler students could do the work by themselves and complete this lesson as independent study. If you choose to have them do this, re-

iew in the latter part of the class period the ideas f complete factorization, the unique factorizaion property, and the greatest common factor.

Have your abler students consider the followng:

1 What is the least number that has 6 and 70 as actors? [210] That has 17 and 102 as factors? 102] That has 12 and 30 as factors? [60]

2 What is the least number that has x and $2x$ as actors? U = C. [$2x$] What is the least number hat has ab and bc as factors when a and c have o common factors? [abc]

For a class of slow learners, use many easy exmples to develop the ideas of complete factorizaion and greatest common factor. Encourage slow earners to use a systematic method for finding omplete factorizations. It will be helpful to reiew with them the set of prime numbers less han 19.

/ Checking up / Pages 304-305

End-of-unit tests

■ Each unit in *STM* contains a set of end-of-unit ests. See page 85 of this *Teaching Guide* for a detailed discussion of the end-of-unit tests and ow to use them.

/ Checking up / Pages 305-307

Cumulative tests

■ Unit 2 and each unit thereafter terminates in a set of cumulative tests. See pages 102 and 103 of his *Teaching Guide* for a detailed discussion of he cumulative tests and how to use them.

7 / 71 Exploring ideas / Pages 308-312

Fractions and equivalent fractions

Objectives

■ The students learn that fractions are ordered pairs of natural numbers. They also learn what is meant by equivalent fractions.

Content overview

■ The idea of ordered pairs of numbers is developed in unit 3. The idea of equivalent ordered pairs is developed in unit 4. Both of these ideas are used in this lesson.

In the elementary grades, students worked rather extensively with fractions at a concrete and computational level. At that time, most of them were not ready to think of the distinction between a fraction and a rational number and the symbols used to express each of these ideas. The students computed with what they thought of as "fractions." Now that the students have learned certain elementary ideas about sets and are familiar with the properties of a number system, they are prepared to study more sophisticated ideas about numbers.

In lesson 72, the idea of a rational number of arithmetic is developed. In succeeding lessons, the properties of the set of rational numbers of arithmetic are developed. The number system involving this set is generalized in lesson 82. More properties and the use of the rational numbers of arithmetic are considered in the latter part of this unit.

The primitive idea of a fraction is the abstraction of thinking of a specific number of parts of a physical object together with the number of parts of the same size into which the object is thought of as being separated.

A *fraction* is an ordered pair of numbers. When interpreted in a physical situation, the second component of the ordered pair, the *denominator* of the fraction, is the number of parts of the same size into which an object is thought of as being separated. The first component of the ordered pair, the *numerator* of the fraction, is the number of these parts that are being considered. The symbol for a fraction is a fraction numeral.

In the physical world, the idea of a fraction involves a definite basic unit. This basic unit is exactly one object. For example, if a certain board is sawed into two pieces of the same size, each piece is $\frac{1}{2}$ of the original board. If another board of the same size as the original board is sawed into four pieces of the same size, two of the pieces represent $\frac{2}{4}$ of the original board. In each case, the original board, or one like it, is the basic unit for the fractions. When the fractions $\frac{1}{2}$ and $\frac{2}{4}$ are considered, it is obvious that $\frac{1}{2}$ of a board is not identical with $\frac{2}{4}$ of a board. However, the fractions $\frac{1}{2}$ and $\frac{2}{4}$ represent, in this case, the same amount of wood. These two fractions are, then, equivalent. Mathematically, two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if and only if $ad = bc$ for the same replacements. The universe for a and c is N, and the universe for b and d is C.

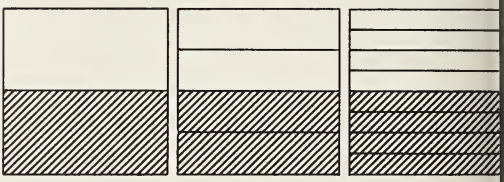
You know that each of the fractions $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$, $\frac{100}{200}$, and so on, represents the same amount of the basic unit. The infinite set of equivalent fractions, each of which can be used to represent the same amount of a basic unit, is a rational number of arithmetic. The set $\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots\}$ is the rational number* $\frac{1}{2}$. While two fractions may be equivalent, two rational numbers may be equal; that is, the rational number $\frac{1}{2}$ is equal to the rational number $\frac{50}{100}$ because the fractions $\frac{1}{2}$ and $\frac{50}{100}$ both indicate the same set.

* Unless otherwise indicated hereafter in this *Teaching Guide*, the term "rational number" is an abbreviation for "rational number of arithmetic."

Instruction notes

■ Before teaching this lesson, you might want t prepare several sheets of paper to help develo the idea of a fraction. Use a felt-tip marking pe to rule off each sheet into parts of the same siz Use colored pencils to shade some of the part Transparent tape may be used to fasten th papers to the chalkboard. When you teach th lesson, use the sheets you have prepared and fo low the procedure used in exercises A through on pages 308 and 309, or use a similar teachin procedure.

After exercise R on page 310, use three sheet of paper that have been ruled off and shaded a follows:



Show these sheets to your class and have th students name the fraction that represents th shaded portion of each sheet. Then ask: "Doe each of the fractions $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{4}{8}$ represent th shaded parts of these sheets?" [Yes] "Are th shaded amounts all the same size?" [Yes] "Eac of these fractions represents a shaded amoun that is the same size as each of the other shade amounts. These fractions are equivalent to eac other. Now you will learn a way to find whethe or not two fractions are equivalent." Continu with exercise A on page 310.

Provision for individual differences

■ Have the abler students consider the following

1 Is $7/7 \sim 6/6$? [Yes] How do you know [$7 \times 6 = 6 \times 7$.]

2 Is $x/x \sim y/y$ a property of counting numbers [Yes] How do you know? [$xy = yx$ symbolize the commutative property of multiplication.]

3 Make replacements in $x/y \sim y/x$ so that you obtain a true statement. U = C. [$6/6 \sim 6/6$ is one example.]

4 How must the replacement for x be related to the replacement for y to obtain a true statement from $x/y \sim y/x$? [The same replacement must be made for x as for y.]

Have your slow learners rule and shade 3″ x 5″ index cards to represent the fractions $\frac{4}{5}$, $\frac{8}{10}$, $\frac{1}{3}$, $\frac{2}{6}$, and $\frac{4}{12}$. On the unruled side of each card they could write two names for the fraction that represents the part-whole relation indicated by the card. For example, on one card they would write "(4, 5), or $\frac{4}{5}$." They will need help with tenths, sixths, twelfths, and so on.

Slow learners may have difficulty with the ideas in this lesson. Frequent review of these ideas, using objects or drawings as visual aids, may be helpful.

/ 72 Exploring ideas / Pages 312-317

Rational numbers

Objectives

The students study the reduction property of fractions. They also learn to define a rational number of arithmetic as a set of equivalent fractions.

Content overview

If the numerator and the denominator of a given fraction have a common factor, they may each be divided by that common factor to obtain a fraction that is equivalent to the given fraction. When the universe for a is N and the universe for b and k is C, a true statement is obtained from $\frac{ca}{cb} \sim \frac{a}{b}$. If the numerator and the denominator of a given fraction are each multiplied by the same counting number, the fraction obtained is equivalent to the given fraction; that is, $\frac{a}{b} \sim \frac{ka}{kb}$.

Two numbers are relatively prime when the intersection of the sets of their factors is {1}. For example, the set of factors of 8 is {1, 2, 4, 8}. The set of factors of 9 is {1, 3, 9}. Since the intersection of these two sets of factors is {1}, 8 and 9 are relatively prime. Notice that relatively prime numbers need not be prime numbers when considered individually.

A fraction whose numerator and denominator are relatively prime is a basic fraction.

A rational number of arithmetic is a set that contains a basic fraction and every fraction that is equivalent to the basic fraction.

Consider the distinction between a fraction and a rational number of arithmetic in a given situation. Suppose that a sheet of paper is separated into three parts of the same size and two of the parts are shaded. The fraction $\frac{2}{3}$ represents both the number of parts into which the sheet is separated and also the number of those parts that are shaded. This situation is different from a situation in which a similar sheet is separated into six parts of the same size and four of the parts are shaded. While the fractions $\frac{4}{6}$ and $\frac{2}{3}$ are related in a special way, they represent two different ideas.

The amount of paper that is shaded in the sheet for which the fraction $\frac{2}{3}$ is used is the same as the amount of paper for which the fraction $\frac{4}{6}$ is used. The amount of paper that is shaded corresponds to the rational number of arithmetic. The rational number is the set that includes the fraction $\frac{2}{3}$ and all fractions equivalent to $\frac{2}{3}$. Each fraction equivalent to $\frac{2}{3}$ represents a different way to think about the *same amount* of paper.

The fractions $\frac{2}{3}$ and $\frac{4}{6}$ are equivalent by the definition of equivalent fractions. Each represents a different idea. But the rational numbers $\frac{2}{3}$ and $\frac{4}{6}$ are equal because they are the same idea; each represents the same amount of a piece of paper that is shaded.

While a fraction is an ordered pair whose components are a numerator and a denominator, a rational number is an infinite set and does *not* have a numerator or a denominator. The rational number $\frac{3}{4}$ does not have a numerator or a denominator because, in this case, the symbol $\frac{3}{4}$ does not name an ordered pair of numbers; it names the set of which the ordered pair (3, 4) is a member.

A rational number of arithmetic is, then, an equivalence class. The proportional relations developed in unit 4 are also equivalence classes.

Although the reduction property of rational numbers is introduced in this lesson, attention to the use of this property is not emphasized at this point. Nor is the use of this property emphasized in the lessons on computation. Since a rational number is a set of equivalent fractions, any one of the fractions may be used to express the rational number. For many purposes in mathematics it is not necessary that answers be expressed as basic fractions.

Instruction notes

■ After exercise G on page 314, ask: "Because a true statement is always obtained from $\frac{a}{b} \sim \frac{ka}{kb}$, do you know that this condition is a property of fractions?" [Yes] "You used the definition of equivalent fractions to help prove this property. What properties of natural numbers did you use to prove this property?" [Associative property of multiplication; commutative property of multiplication]

After exercise O on page 315, write the numerals $\frac{6}{17}$ and $\frac{8}{20}$ on the chalkboard. Ask: "What is the set of factors of 6?" [{1, 2, 3, 6}] "Of 17?" [{1, 17}] "What is the intersection of these two sets?" [{1}] "Are 6 and 17 relatively prime numbers?" [Yes] "Is $\frac{6}{17}$ a basic fraction?" [Yes] "What is the set of factors of 8?" [{1, 2, 4, 8}] "Of 20?" [{1, 2, 4, 5, 10, 20}] "What is the intersection of these two sets?" [{1, 2, 4}] "Are 8 and 20 relatively prime?" [No] "Suppose that the intersection of the sets of factors of two numbe contains only the number 1. Are the two numbe relatively prime?" [Yes]

Before exercise L on page 317, say: "Yo know that a rational number of arithmetic is a infinite set. Does a set have either a numerator o a denominator?" [No] Write the numeral $\frac{3}{4}$ o the board. Say: "When you think of the fractio named by this symbol, you think of an ordere pair of numbers. When you think of the ration number of arithmetic named by this symbol, yo think of the set of which the fraction $\frac{3}{4}$ is a mem ber. From now on, whenever you see a symb like this, you should think of a rational numbe of arithmetic. When a symbol like this is used t name a fraction, the book will always tell yo that you are to think of a fraction."

Provision for individual differences

■ A class of abler students could omit the follow ing developmental exercises:

D, page 312
G, I, page 313
G through J, R through U, page 314
F, G, L, M, page 315
B, D, page 316
G, H, page 317

For ungrouped, average, and slow classes, yo may find it necessary to devote two periods t this lesson. If so, in the first period, complet through exercise O on page 315. Assign "On you own" exercises 1 through 10. Finish the lesson o the second day and assign the remaining "O your own" exercises.

Have the slow learners work with the fractio cards they made for the last lesson. They coul group the cards for $\frac{4}{5}$, $\frac{8}{10}$, and $\frac{12}{15}$ together to show that these fractions are members of the same ra tional number of arithmetic. It may be difficul for slow learners to grasp the idea of a rationa number. Their previous teachers may have s thoroughly indoctrinated them with traditiona

deas about fractions that some students find it oo difficult to "unlearn" what has been taught efore. For example, it has been customary for rithmetic teachers to insist that students "reduce ractions to lowest terms." This insistence may ave colored the student's thinking to the extent hat he can view a rational number only in terms f a basic fraction. Thus, it may be difficult for ome slow learners to accept the fraction numerals $\frac{4}{8}$ and $\frac{1}{2}$ as representing the same rational umber.

To reduce the teaching time needed for this esson, you can omit that part of the lesson dealng with a proof for the reduction property of ractions, that is, display 3 and developmental exrcises A through M on pages 313 and 314.

/ 73 Exploring ideas / Pages 318-322

Ordering of the rational numbers

Objectives

The students learn to associate the rational numbers of arithmetic with points in a line. They also learn to use the idea of "less than" in ordering the rational numbers.

Content overview

The symbol R_a, read "the set of rational numbers of arithmetic," is introduced in this lesson. The members of R_a do not have positive or negative signs. Observe that R_a is an infinite set and that each member of R_a is an infinite set of equivalent fractions. Obviously, the idea of a rational number of arithmetic is more sophisticated than the idea of a fraction.

The development of R_a is necessary because some problems cannot be solved if the universe is the set of natural numbers. R_a, together with two operations and certain properties, forms the system of rational numbers of arithmetic. This development is generalized in lesson 82.

A rational-number line is a line in which rational numbers are associated with points. When we say "rational-number line" in the text, we mean "rational-number-of-arithmetic line." To make a rational-number line, first assign a point for $\frac{0}{1}$ on a picture of a line. Then assign a point for $\frac{1}{1}$ to the right of point $\frac{0}{1}$. Use the segment determined by points $\frac{0}{1}$ and $\frac{1}{1}$ to locate points $\frac{2}{1}$, $\frac{3}{1}$, and so on. Locate the points for $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, and $\frac{4}{5}$ by separating the segment determined by points $\frac{0}{1}$ and $\frac{1}{1}$ into five congruent segments. Assign $\frac{1}{5}$ to the endpoint at the right of the first segment, $\frac{2}{5}$ to the endpoint at the right of the second segment, and so on.

In general, we locate the point for $\frac{a}{b}$ by separating the segment determined by $\frac{0}{1}$ and $\frac{1}{1}$ into b congruent segments. Also, we locate points to the right of point $\frac{1}{1}$ to determine segments that are each congruent to a $\frac{1}{b}$ segment. Assign $\frac{a}{b}$ to the endpoint at the right of the ath segment.

The rational numbers $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, and so on, are all associated with exactly one point because the fractions $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{3}{6}$ indicate the same number.

While every rational number is associated with a point in a line, not every point in a line is associated with a rational number. Thus, a rational-number line has "empty spaces" that are associated with the set of irrational numbers. Examples of irrational numbers are π and $\sqrt{2}$.

There is an infinite set of rational numbers between any two given rational numbers. This property, called the "density property of rational numbers," is presented in lesson 89. The natural numbers do not have this property, but the irrational numbers, like the rationals, have the density property.

Note that the natural number 1 is not the same number as the rational number $\frac{1}{1}$. We associate

the natural number 1 with the set $\{0\}$. We associate the rational number $\frac{1}{1}$ with the set $\{\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \ldots\}$. The rational number $\frac{2}{2}$ is in the same set as the rational number $\frac{1}{1}$. 1 is a member of N; $\frac{1}{1}$ is a member of R_a.

Observe how a rational-number line is used to order members of R_a. If $\frac{a}{b}$ is between $\frac{0}{1}$ and $\frac{c}{d}$, then $\frac{a}{b} < \frac{c}{d}$. For the number line pictured in D4, $\frac{a}{b} < \frac{c}{d}$ if point $\frac{a}{b}$ is to the left of point $\frac{c}{d}$. You can use this method to determine if one given rational number is less than another given rational number only when you have located the points in the line that correspond to the rational numbers under consideration.

There is a "less than" property for rational numbers. If $\frac{a}{b} < \frac{c}{d}$, then $ad < cb$ for the same replacements. The universe for a and c is N. The universe for b and d is C.

Because of the relation of the ideas of "greater than" and "less than" and because of the "less than" property for rational numbers, if $\frac{a}{b} > \frac{c}{d}$, a true statement is also obtained from $ad > cb$.

Instruction notes

■ After the paragraph on page 318 that introduces the symbol R_a, you may want to use pages 50 and 51 to review the procedures you used to locate points in a natural-number line.

As you develop the ideas in exercises G through J on pages 318 and 319, sketch on the chalkboard a picture similar to that in D2.

Before exercise R on page 319, say: "Suppose that we wanted to locate the points for $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, and so on. If point $\frac{0}{1}$ and $\frac{1}{1}$ were located, how could we locate points for fifths?" [Separate the segment determined by points $\frac{0}{1}$ and $\frac{1}{1}$ into five congruent segments. Assign $\frac{1}{5}$ to the right endpoint of the first segment, $\frac{2}{5}$ to the right endpoint of the second segment, and so on.]

After exercise C on page 320, write the sentence $\frac{17}{21} < \frac{10}{13}$ on the board. Ask: "Are either of points $\frac{17}{21}$ or $\frac{10}{13}$ represented by dots in D4?" [No] "Do you know for certain which of these point is to the left of the other?" [No] "We need an other way to find if one rational number of arith metic is less than another rational number arithmetic."

After exercise U on page 321, say: "Loo again at exercise M. Does each of the rationa numbers expressed contain a fraction with a de nominator of 12?" [Yes] Write the sentenc $\frac{16}{12} < \frac{15}{12}$ on the board. Ask: "Does this sentence ex press the same statement as sentence M?" [Yes "Each rational number expressed in exercise contains a fraction with what common denomi nator?" [59×40, or 2360] "You know that th 'common denominator' method can be used. D you think it involves more computation than th use of the definition of 'less than'?" [Yes]

Provision for individual differences

■ The abler students could consider the exercise below. For each variable, U = C.

1 If $b < c$, how is $\frac{b}{a}$ related to $\frac{c}{a}$? $[\frac{b}{a} < \frac{c}{a}.]$

2 If $b < c$, how is $\frac{a}{b}$ related to $\frac{a}{c}$? $[\frac{a}{b} > \frac{a}{c}.]$

3 How is $\frac{a}{b}$ related to $\frac{a+c}{b}$? $[\frac{a}{b} < \frac{a+c}{b}.]$

4 How is $\frac{a}{b}$ related to $\frac{a}{b+c}$? $[\frac{a}{b} > \frac{a}{b+c}.]$

5 How is $\frac{2}{3}$ related to $\frac{2+5}{3+5}$? $[\frac{2}{3} < \frac{2+5}{3+5}.]$

6 What other condition must be satisfied to ob tain a true statement from $\frac{a}{b} < \frac{a+c}{b+c}$? $[a < b.]$

7 What other condition must be satisfied to ob tain a true statement from $\frac{a}{b} > \frac{a+c}{b+c}$? $[a > b.]$

For an average or ungrouped class, you may find it necessary to devote two periods to this les son. In the first period, complete through exer cise W on page 320 and assign "On your own" exercises 1 through 14.

Provide extra practice on the use of the "less than" property for slow learners. These students should not be expected to master the definition o "less than" using variables. The idea involved is the important thing. For students who lack skil in computation, you may find it helpful to elimi nate or replace exercises in the "On your own"

vork. Exercises 20, 21, 25, and 26 may be difficult or the slow learners.

/ Special challenge / Pages 323-324

Objectives

■ The students consider a topic in number theory: "Is there a greatest prime number?"

Content overview

■ If the set of prime numbers is a finite set, then here is a greatest prime number. If it can be proved that there is no greatest prime number, hen the set of primes is an infinite set.

We assume that the set of primes (P) is finite. We can order this set by using the idea of "less han" and can symbolize the set as $P = \{p_1, p_2, p_3, \ldots, p_k\}$. p_1 is the least prime; p_2 is the second-least prime; and so on.

We define c as the counting number that is the sum of 1 and the product of all the prime numbers in P. If c is a prime number, it is greater than p_k. If c is a composite number, each prime factor of c is greater than p_k. We conclude that there is no greatest prime number, and, therefore, the set of primes is an infinite set.

We use p_k to represent the greatest prime number. Observe that the letter p is not in italics in the student's book because it is not used to express a variable in this set of exercises.

7 / 74 Exploring ideas / Pages 324-327

Multiplication of rational numbers

Objectives

■ The students learn how to find the product of two rational numbers of arithmetic. They also study the closure property of multiplication and the well-defined property of multiplication of rational numbers of arithmetic.

Content overview

■ The rational-number system includes two operations (addition and multiplication) on pairs of members of R_a. The operation of multiplication is developed first because it is easier to define the product of two rational numbers than to define the sum of two rational numbers of arithmetic.

Multiplication of rational numbers of arithmetic is a many-to-one mapping of the members of $R_a \times R_a$ onto the members of R_a. The rule used in the mapping defines the product of two rational numbers. Multiplication is, of course, a binary operation.

The rule for finding the product of two rational numbers is $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$. For a and c, U = N; for b and d, U = C. This definition is developed on page 325.

Notice that multiplication as taught in this lesson involves rational numbers. Remember that a rational number of arithmetic is a set of equivalent fractions. Thus, multiplication of a member of R_a involves infinite sets of fractions and is *not* an operation performed on numerators and denominators of fractions. When the definition of the product of two rational numbers is used, fractions that indicate the rational numbers are used to obtain a fraction that indicates the product. Note that the product is a rational number.

While the binary operation of multiplication is a many-to-one mapping of pairs of rational numbers onto rational numbers, the process of using the definition of the product of two rational numbers is performed at the computational level. So it is not really essential that your students grasp all the subtleties of the distinction that we are making between rational numbers and fractions. At least at this point in their mathematical devel-

opment, they can go ahead without making a sharp distinction between the set that we have called a rational number and a member of this set that we called a fraction.

Multiplication of rational numbers is an operation, and as such it has certain properties. Two of these properties are developed in this lesson. The closure property is developed first. Although the closure property of multiplication of natural numbers has been discussed in previous lessons, the closure property of the operation of multiplication of rational numbers is a different property and must be developed for the students. The well-defined property of multiplication of rationals is also developed in this lesson.

The term "well defined" may cause some difficulty. The well-defined property of multiplication of rational numbers may be explained in this way: if $ab = x$, $a = c$, and $b = d$, then $cd = x$. $U = R_a$. We say the product ab is well defined. When the $\frac{1}{2}$ in $\frac{1}{2} \times \frac{3}{4}$ is replaced by $\frac{2}{4}$ to obtain $\frac{2}{4} \times \frac{3}{4}$, the product is indicated by the fraction $\frac{6}{16}$ rather than by the fraction $\frac{3}{8}$. But, as it turns out, $\frac{6}{16}$ and $\frac{3}{8}$ indicate the same rational number. This always happens. Either number or both numbers used to form a product may be replaced by equal numbers. $(\frac{1}{2}, \frac{3}{4})$ and $(\frac{2}{4}, \frac{3}{4})$ each map onto the same rational number. We use this property in proofs and also as a computational aid.

Multiplication of rational numbers also has a commutative and an associative property. The latter two properties are developed in lessons 80 and 81.

Instruction notes

■ Before you begin the lesson, write on the chalkboard:

(6, 4)	(3, 8)	(12, 2)
↓	↓	↓
6×4, or 24	3×8, or 24	12×2, or 24

Say: "You know that multiplication of natural numbers is a binary operation. Is it a many-to-one mapping?" [Yes] Ask: "Why is it called a many-to-one mapping?" [Two or more ordered pairs are mapped onto the product.] "What rule do we use to find the number that two given natural numbers map onto by multiplication?" [The definition of the product of two natural numbers]

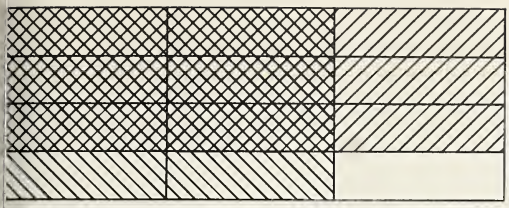
Write on the chalkboard:

$$(a, b)$$
$$\downarrow$$
$$c$$

Ask: "If the universe for each of these variables is N, is there always a natural number onto which a pair of natural numbers is mapped by multiplication?" [Yes] "In this lesson you will discover if ordered pairs of rational numbers of arithmetic map onto members of R_a by multiplication."

Begin the developmental exercises. Before exercise E on page 325, say: "You know that a product of two rational numbers is a rational number that is indicated by a fraction. But do you really multiply fractions and obtain a product that is a fraction?" [No] "Now you see that a product is a number. A fraction is an ordered pair of numbers." Write the sentence $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$ on the chalkboard. Then make a sketch like the following picture:

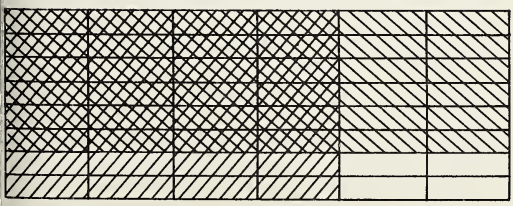
Ask: "Does this sketch represent three fourths of an object?" [Yes] Draw more lines so that the picture looks like the following:



Say: "Now I have also indicated two thirds of the object. Into how many pieces of the same size have I separated the object?" [12] "How many of the pieces have double shading?" [6] "Is $\frac{2}{3}$ times equal to $\frac{6}{12}$?" [Yes] "This is an example of the way in which mathematical ideas often are suggested by the world of things. But later we treat them as purely mathematical ideas, expressed by definitions and rules." Do not erase the picture on the board.

After exercise G on page 326, say: "Let us review what we mean by equal numbers. Can two different numbers be equal?" [No] "The product 2×3 is equal to 6 because 2×3 *is* the number 6. $\frac{8}{2}$ is equal to $\frac{2}{3}$ because they are the same rational number, even though they are not indicated as the same fraction. Is the rational number $\frac{8}{12}$ exactly the same rational number as $\frac{2}{3}$?" [Yes] "Remember that one number is equal to a second number only when both are the same number."

After exercise T on page 326, write the sentence $\frac{4}{6} \times \frac{6}{8} = \frac{24}{48}$ on the chalkboard. Change the picture you drew earlier so that it looks like the following:



Say: "This picture now represents $\frac{4}{6}$ times $\frac{6}{8}$. Into how many pieces of the same size is the object now separated?" [48] "How many of the pieces have double shading?" [24] "Is $\frac{4}{6}$ times $\frac{6}{8}$ equal to $\frac{24}{48}$?" [Yes] "Is $\frac{4}{6}$ the same number as $\frac{2}{3}$?" [Yes] "Is $\frac{6}{8}$ the same number as $\frac{3}{4}$?" [Yes] "Is $\frac{24}{48}$ the same number as $\frac{6}{12}$?" [Yes] "Is this drawing an example of the use of the well-defined property of multiplication?" [Yes]

Provision for individual differences

■ The following developmental exercises could be omitted by a class of abler students:

B, page 324

E, H, K, L, page 325

I, M through Q, page 326

Abler students may consider the exercises below.

Arrange each group of numbers named in exercises 1 through 4 according to the idea of "less than."

1 $\frac{1}{2}, \frac{1}{3}, \frac{1\times1}{2\times3}$ [$\frac{1\times1}{2\times3}, \frac{1}{3}, \frac{1}{2}$]

2 $\frac{2}{3}, \frac{3}{5}, \frac{2\times3}{3\times5}$ [$\frac{2\times3}{3\times5}, \frac{3}{5}, \frac{2}{3}$]

3 $\frac{5}{3}, \frac{3}{2}, \frac{5\times3}{3\times2}$ [$\frac{3}{2}, \frac{5}{3}, \frac{5\times3}{3\times2}$]

4 $\frac{6}{3}, \frac{1}{2}, \frac{6\times1}{3\times2}$ [$\frac{1}{2}, \frac{6\times1}{3\times2}, \frac{6}{3}$]

5 Which of the exercises above shows that the product of two rational numbers may be greater than either rational number? [3]

6 Which of the exercises show that the product of two rational numbers may be less than either rational number? [1 and 2]

7 If $\frac{a}{b} < \frac{1}{1}$ and $\frac{c}{d} < \frac{1}{1}$, how is the product $\frac{a}{b}(\frac{c}{d})$ related to $\frac{a}{b}$? [$\frac{a}{b}(\frac{c}{d}) < \frac{a}{b}$.] To $\frac{c}{d}$? [$\frac{a}{b}(\frac{c}{d}) < \frac{c}{d}$.] To $\frac{1}{1}$? [$\frac{a}{b}(\frac{c}{d}) < \frac{1}{1}$.]

8 If $\frac{a}{b} < \frac{1}{1}$ and $\frac{c}{d} > \frac{1}{1}$, how is the product $\frac{a}{b}(\frac{c}{d})$ related to $\frac{a}{b}$? [Greater than $\frac{a}{b}$] To $\frac{c}{d}$? [Less than $\frac{c}{d}$] To $\frac{1}{1}$? [Depending on the replacements made, the product will be less than, equal to, or greater than $\frac{1}{1}$.]

For your slow learners, use many additional examples to illustrate each of the properties developed in the lesson. Replace some of the difficult exercises in the "On your own" exercises with some you consider more suitable for slow learners.

You could discuss exercises 1 through 8; then have a student work one of these exercises at the board; and then let the class discuss the exercise. Before assigning exercises 24 through 29, be sure to discuss the example (exercise 24) thoroughly. Preferably, discuss one or two more.

---

7 / 75 Exploring ideas / Pages 327-332

Addition of rational numbers

Objectives

■ The students learn how to find the sum of two rational numbers. They also study the closure property of addition and the well-defined property of addition of rational numbers.

Content overview

■ The binary operation of addition of rational numbers is a many-to-one mapping of the members of $R_a \times R_a$ onto members of R_a. The rule that is used in the mapping defines the sum of two rational numbers.

The definition of the sum of two rational numbers is simply the assertion that a true statement is always obtained from $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$. The universe for a and c is N, and the universe for b is C. This property is applicable to any two rational numbers, whether or not they are indicated by fractions with the same number for each denominator. When the two rational numbers are indicated by fractions with unequal denominators, you use the well-defined property of addition.

Consider the sum $\frac{1}{3} + \frac{2}{5}$. To use the definition of sum, the rational numbers must be indicated by fractions with equal denominators. Since $\frac{2}{5} = \frac{6}{15}$ and $\frac{1}{3} = \frac{5}{15}$, $\frac{1}{3} + \frac{2}{5} = \frac{5}{15} + \frac{6}{15}$ (using the well-defined property of addition). $\frac{5}{15} + \frac{6}{15} = \frac{5+6}{15}$, or $\frac{11}{15}$.

From working with examples such as the one described in the last paragraph, we can make the generalization $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$. This condition is the sum property of rational numbers.

An alternative method of developing the sum of two rational numbers is first to define the sum as $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$. Then, for the special case in which the replacement for d is the same number as the replacement for b, the following steps may be taken:

1 The sum is defined this way: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

2 According to the definition: $\frac{a}{b} + \frac{c}{b} = \frac{ab+bc}{b(b)}$.

3 The distributive property of multiplication over addition of natural numbers is used to obtain: $\frac{a}{b} + \frac{c}{b} = \frac{b(a+c)}{b(b)}$.

4 The reduction property of fractions is used to obtain: $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$.

The authors considered this alternative method and discarded it in favor of the method used in the lesson. Because it is easier for students to understand the simpler condition, this simpler condition was used in the definition of the sum, although this means that students must learn both conditions.

The well-defined property of addition of rational numbers of arithmetic is essential. That is, the definition of the sum of two rational numbers must yield the same sum when one, or both, of the rational numbers is replaced by an equal rational number.

Because of the closure property of addition of natural numbers, the sum $a + c$ is a natural number. Since b is a counting number, the sum $\frac{a+c}{b}$ is a rational number that is indicated by a fraction whose numerator is a natural number and whose denominator is a counting number.

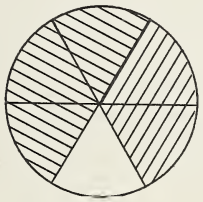
Observe that, even though the term "common denominator" is used in the text, no mention is made of a least common denominator. The use of a least common denominator is a device for simplifying computation. But the students do not need this idea yet. We develop the idea of least common multiple and use it to introduce the tra-

litional "least common denominator" in lesson 77.

Instruction notes

Begin the lesson by reviewing with your class he idea that addition of natural numbers is a inary operation that is a many-to-one mapping f the members of $N \times N$ onto the members of N. Then ask: "What rule do you use for the addiion mapping of an ordered pair of natural numbers onto a natural number?" [The definition of he sum of two natural numbers]

Before exercise C on page 327, write the sentence $\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$ on the chalkboard. Ask: "What dea is expressed by this sentence?" [The sum $\frac{3}{6}$ plus $\frac{2}{6}$ is the rational number $\frac{5}{6}$.] Sketch the following on the board:



Ask: "Is $\frac{3}{6}$ of an object represented by the amount that is shaded vertically? Is $\frac{2}{6}$ of that object represented by the amount that is shaded horizontally? What amount of the object is shaded?" [$\frac{5}{6}$] "Is $\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$ a true statement?" [Yes]

In discussing exercises A through D on page 329, try to develop the idea that addition of rational numbers of arithmetic is defined in a way that ensures closure for this operation.

After exercise G on page 329, remind the students that when we say that $\frac{28}{14}$ and $\frac{4}{2}$ are equal, we mean that they are the same number.

Before exercise C on page 330, say: "Suppose that you are thinking about a given fraction that indicates a rational number of arithmetic. If you use the reduction property of fractions to obtain a fraction equivalent to the given fraction, does the fraction you obtain always indicate the same rational number as the given fraction?" [Yes]

At the end of this lesson, you might have your students make a list of the definitions and properties they have studied so far in this unit.

Provision for individual differences

■ A class of abler students could omit the following developmental exercises:

O through U, page 328
K, L, P, S, page 330
I through M, page 331

Have your abler students consider the exercises below.

Use the sum property to find what member of R_a each ordered pair of rational numbers expressed in exercises 1 through 6 is mapped onto by addition.

1 $(\frac{1}{5}, \frac{1}{4})$ $[\frac{9}{20}]$
2 $(\frac{0}{8}, \frac{2}{5})$ $[\frac{16}{40}]$
3 $(\frac{0}{b}, \frac{c}{d})$ $[\frac{bc}{bd}]$
4 $(\frac{1}{b}, \frac{1}{d})$ $[\frac{d+b}{bd}]$
5 $(\frac{0}{b}, \frac{0}{d})$ $[\frac{0}{bd}]$
6 $(\frac{a}{b}, \frac{a}{b})$ $[\frac{2ab}{b^2}]$

7 When is a true statement obtained from $\frac{a}{b}=\frac{ad+bc}{bd}$? [When c is replaced by 0]

8 When is a true statement obtained from $\frac{a}{b}>\frac{ad+bc}{bd}$? [Never, if the universe for c is N]

Abler students should understand the significance of the definition of the sum of rational numbers and the properties relating to this definition. That is, the addition of rational numbers is so defined that this operation has the closure property; it is well defined; and it has the sum property. The definition, along with the properties, enables students to find the sum of any pair of the same or different rational numbers.

For an ungrouped, average, or slow class, you may prefer to devote two periods to this lesson. On the first day, complete the lesson through the two paragraphs that follow exercise J on page 329. Assign "On your own" exercises 1 through 8, and exercises 10, 14, 16, 19, 22, 27, and 32.

Give your slow learners extra practice in the use of the sum property. Slow learners may have

considerable difficulty in relating the mechanical work of addition using fraction numerals to the abstract definition for addition and the abstract representation of the properties. It may be more meaningful to develop the reasons why they can work with $\frac{1}{4}+\frac{2}{4}$ and similar examples and then move on to examples such as $\frac{1}{3}+\frac{2}{5}$ without emphasizing the definitions involving variables.

7 / 76 Exploring ideas / Pages 332-335

Subtraction of rational numbers

Objectives

■ The students learn to relate subtraction of rational numbers of arithmetic to the operation of addition and to find the difference of two rational numbers of arithmetic.

Content overview

■ We know that subtraction of natural numbers is related to the operation of addition because $a-b=x$ also means $a=b+x$. Subtraction of members of R_a is related to addition in the same way. For example, $\frac{3}{4}-\frac{1}{3}=\frac{x}{12}$ means $\frac{3}{4}=\frac{1}{3}+\frac{x}{12}$. This relationship is used to develop an understanding of the definition of the difference of two rational numbers of arithmetic.

The definition of subtraction of rational numbers of arithmetic does not give subtraction the closure property. Hence, we do not think of subtraction of these numbers as an operation. We avoided this issue in the student's book. We considered it unnecessary, and we thought it might confuse a student who had previously met subtraction only in computation and had learned to think of it as an operation.

After the student has been introduced to the set of rational numbers in *STM 2*, he will learn to think of subtraction as the addition of an additive inverse. That is, $a-b=a+(\bar{\ }b)$. Obv ously, we could not introduce this idea in *STM* At this point a student can think of the differen of two members of R_a only in terms of the rel tionship between subtraction and addition of r tional numbers of arithmetic.

The definition of the difference of tw members of R_a is expressed by the senten $\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$. A true statement is always obtaine from this condition when the replacement for a equal to or greater than the replacement for The universe for b is C, and the universe for and c is N. When the replacement for c is greate than the replacement for a, the difference $\frac{a}{b}-\frac{c}{b}$ not a rational number of arithmetic. Thus, R_a not closed under subtraction.

The difference property of rational numbers arithmetic is proved by using the definition of th difference of two rational numbers of arith metic and the well-defined property of additio $\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$ is the definition of difference. B cause equivalent fractions indicate the sam member of R_a, $\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}$ follows fro the well-defined property. Then, by definitio $\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd}$. Using the transitive property equality, we obtain the property $\frac{a}{b}-\frac{c}{d}=\frac{ad-b}{bd}$ This sentence expresses the difference property rational numbers of arithmetic. To obtain a tru statement from the difference property, the pro uct ad cannot be less than the product bc.

Instruction notes

■ You may want to teach the first part of thi lesson without using the first few developmenta exercises. If you choose to do this, write o the chalkboard: "$15-8=7$. $15=8+7$." Say "Does $15-8=7$ mean $15=8+7$?" [Yes] "Sub traction has the same relation to addition in th set of rational numbers of arithmetic." Writ these sentences on the board: $\frac{10}{3}-\frac{9}{3}=\frac{1}{3}$. $\frac{10}{3}=\frac{9}{3}+\frac{1}{3}$ Say: "For example, $\frac{10}{3}-\frac{9}{3}=\frac{1}{3}$ means $\frac{10}{3}=\frac{9}{3}+\frac{1}{3}$." Write the sentence $\frac{11}{13}-\frac{7}{13}=\frac{x}{13}$ on the board. Ask

What does this condition mean?" [$\frac{11}{13} = \frac{7}{13} + \frac{x}{13}$.] What is the solution?" [4] Begin the developmental exercises with exercise E on page 332.

After exercise H on page 333, write the sentence $\frac{7}{3} - \frac{2}{3} = \frac{5}{3}$ on the chalkboard. Sketch the following:

Ask: "Does this picture represent $\frac{7}{3}$ of an object?" [Yes] Shade any two of the thirds. Say: "Is the shaded amount two thirds of the object?" [Yes] "What is the amount that is not shaded?" [$\frac{5}{3}$] Write the sentence $\frac{7}{3} = \frac{2}{3} + \frac{5}{3}$ on the board. Say: "You know that the picture represents $\frac{7}{3}$ of an object. Do the shaded amount and the unshaded amount together represent $\frac{7}{3}$ of the object?" [Yes] "The sum $\frac{2}{3}$ plus $\frac{5}{3}$ is what rational number of arithmetic?" [$\frac{7}{3}$]

Before exercise B on page 334, ask: "Are parentheses really needed around the symbols 2×4 and 3×1?" [No] "Why not?" [When simplifying an expression, you should always multiply before you subtract.] Write: $\frac{2(4) - 3(1)}{3(4)}$. Ask: "Could this expression also be used?" [Yes]

Provision for individual differences

A class of abler students could omit the following developmental exercises:

M, N, P through T, page 333

H, J, K, L, page 334

Abler students could consider the following exercises:

Use the difference property to find the difference of the first and second components of each ordered pair of rational numbers expressed.

1 $(\frac{9}{10}, \frac{9}{3})$ $[\frac{27}{30}]$ **2** $(\frac{a}{b}, \frac{a}{d})$ $[\frac{ad}{bd}]$ **3** $(\frac{1}{1}, \frac{2}{4})$ $[\frac{2}{4}]$ **4** $(\frac{a}{b}, \frac{a}{b})$ $[\frac{0}{b^2}]$ **5** $(\frac{1}{b}, \frac{1}{d})$ if $d > b$. $[\frac{d-b}{bd}]$ **6** $(\frac{1}{1}, \frac{c}{d})$ if $d > c$. $[\frac{d-c}{d}]$

For exercises 7, 8, and 9, tell what must be true of $\frac{c}{d}$ if a true statement is obtained.

7 $\frac{a}{b} - \frac{c}{d} = \frac{a}{b}$. $[c = 0.]$

8 $\frac{a}{b}(\frac{c}{d}) = \frac{a}{b}$. $[c = d \text{ and } d \neq 0.]$

9 $\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{c}{d}$. $[c = 0.]$

For slow learners, avoid emphasizing those sentences in which placeholders are used. Rely on the students' previous background and the use of simple examples to develop the ideas in the lesson.

7 / 77 Exploring ideas / Pages 337-341

Common factors and common multiples

Objectives

■ The students learn to use the ideas of greatest common factor and least common multiple as aids in computation.

Content overview

■ Rather than treat each of these computational ideas as a "trick," we develop the mathematical foundation for each idea in the lesson.

We introduced basic fractions in lesson 72. To obtain a basic fraction, you can divide by one common factor of a numerator and a denominator at a time. You can continue this process until the numerator and the denominator are relatively prime. It is often quicker to use a complete factorization of each of the numbers and divide by the product of the common factors. For example, the complete factorization of the numerator and the denominator of the fraction $\frac{36}{96}$ is $\frac{2^2 \times 3^2}{2^5 \times 3^1}$. The common factors of 36 and 96 are 2^2 and 3^1. The product of the common factors is 12, which is the greatest common factor. When each component of $\frac{36}{96}$ is divided by 12, the basic frac-

tion $\frac{3}{8}$ is obtained. Because people often use the greatest common factor as a divisor, they sometimes call it the *greatest common divisor*.

A basic-fraction numeral, for example, $\frac{3}{8}$, is easier to understand than $\frac{108}{288}$, even though both symbols name the same number. Also, it usually is easier to perform operations when the rational numbers of arithmetic are indicated by basic fractions. For example, it is easier to find a standard name for the sum $\frac{7}{5}+\frac{2}{5}$ than to find a standard name for the sum $\frac{49}{35}+\frac{26}{65}$, even though the two sums are the same.

Notice how the ideas just discussed are used on pages 338 and 339 to find the basic fraction of the product of two rational numbers of arithmetic whose numerators and denominators are expressed as the product of their prime factors. Consider $\frac{8}{21}\times\frac{35}{6}=\frac{2\cdot 2\cdot 2}{3\cdot 7}\times\frac{5\cdot 7}{2\cdot 3}$. Because of the definition of the product of two rational numbers, $\frac{2\cdot 2\cdot 2}{3\cdot 7}\times\frac{5\cdot 7}{2\cdot 3}=\frac{2\cdot 2\cdot 2\cdot 5\cdot 7}{3\cdot 7\cdot 2\cdot 3}$. The numerator and the denominator have a common factor of $2\cdot 7$, which, when used as a divisor, yields $\frac{2\cdot 2\cdot 5}{3\cdot 3}$. Therefore, the basic fraction of the product $\frac{8}{21}\times\frac{35}{6}$ is $\frac{20}{9}$.

A multiple of a given number is simply the product of a counting number and the given number. The given number is, then, a factor of each of its multiples.

The *least common multiple* of several given numbers is the least number that is a multiple of these numbers. Although 48 is a common multiple of 6 and 8, the least common multiple of 6 and 8 is 24.

You can find the least common multiple of two numbers by using the complete factorization of the two numbers. The least common multiple is the product of the greatest powers of the prime numbers that are factors of either or both numbers. The complete factorization of 6 is $2^1\times 3^1$ and of 8 is 2^3. The least common multiple of 6 and 8 is $2^3\times 3^1$, or 24. The complete factorization of 24 is $2^3\times 3^1$ and of 36 is $2^2\times 3^2$. The least common multiple of 24 and 36 is $2^3\times 3^2$, or 72.

You can apply the idea of least common multiple to both addition and subtraction of members of R_a. The least common denominator for two or more fractions is the least common multiple of their denominators.

When a standard name of a sum or a difference is to be found, it is often easiest first to see the rational numbers involved are indicated by basic fractions. The least common denominator of the basic fractions is then used as the denominator of the fraction that indicates the sum or the difference. Consider the sum $\frac{7}{42}+\frac{15}{27}$. If basic fractions are used, the sum is $\frac{1}{6}+\frac{5}{9}$. The least common multiple of 6 and 9 is 18. The sum is $\frac{3+10}{18}$, or $\frac{13}{18}$.

We could have developed the computations aids in this lesson in other ways, of course. A least common multiple can be found by testing successive multiples of the greater of two numbers for divisibility by the lesser number. For example, the least common multiple of 8 and could be found by thinking in this way: "Is 1(8 divisible by 6? Is 2(8) divisible by 6? Is 3(8) divisible by 6?" The least multiple of the greater number that is divisible by the lesser number is the least common multiple of the two numbers. Encourage students to find short cuts in computation, but also require them to understand the short cuts in terms of such mathematical ideas as basic fractions and complete factorizations.

Instruction notes

■ Because of the length of the lesson, you could work through developmental exercise V on page 339 and assign "On your own" exercises 1 through 19 on the first day. Then you could finish the lesson and assign the remaining "On your own" exercises on the second day.

Write the sentences $\frac{60}{20}-\frac{24}{12}=x$ and $\frac{3}{1}-\frac{2}{1}=x$ on the chalkboard. Say: "Each of these sentences expresses the same condition. Is the solution

asier to find when basic fractions are used?" Yes]

After exercise V on page 339, write the expression $\frac{9}{13} \times \frac{26}{15}$ on the board. Say: "Suppose we want o find the basic fraction for this product. Can ve divide 26 and 13 each by 13?" [Yes] Draw trokes through the numerals 13 and 26. Write "1" below 13 and "2" above 26. "Can we divide and 15 each by 3?" [Yes] Draw strokes through hose numerals and write "3" above 9 and "5" below 15. Ask: "What is the basic fraction for his product?" [$\frac{6}{5}$] "Upon what mathematical deas is this way of multiplying rational numbers based?" [Definition of product, reduction property, complete factorization, and greatest common factor]

Provision for individual differences

■ A class of abler students could omit the following developmental exercises:

A, B, page 337
M, N, O, page 338
N, S, U, page 339
L, M, N, page 340

Abler students could consider the following exercises:

1 (a) Name three members of $\{(x, y) | x = 3y\}$. $U = C \times C$. [Examples are: (3, 1), (6, 2), (9, 3).]

(b) Is the first component a multiple of the second component in each ordered pair you named? [Yes]

(c) For each member of this set, is the first component a multiple of the second component? [Yes] How do you know? [The first component is the product of 3 and the second component.]

2 (a) If x and y are prime numbers, what is their greatest common factor? [1]

(b) What is their least common multiple? [xy]

3 (a) If x is a factor of y, what is the greatest common factor of x and y? [x]

(b) What is their least common multiple? [y]

Give your slow learners extra practice with exercises in which they find the basic fraction for a rational number, the basic fraction for a rational number that is a product, and the least common multiple of two counting numbers. As you work with slow learners, relate the ideas of greatest common factor and least common multiple to their previous experience in computation with fractions. Emphasize the fact that the least common denominator for two fractions is the same as the least common multiple of their denominators.

7 / 78 Exploring ideas / Pages 342-344

The natural numbers related to certain rational numbers

Objectives

■ The students learn that the set of natural numbers is related to a specific subset of the set of rational numbers of arithmetic.

Content overview

■ This short lesson, which appears to be simple, develops one of the most powerful ideas in mathematics.

The set of natural numbers and $\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \ldots\}$ can be placed in one-to-one correspondence. $\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \ldots\}$, or set S, is a subset of R_a. Observe that this subset is an ordered set, as is the set of natural numbers. Because 5 and $\frac{5}{1}$ are mapped onto each other under this correspondence, we call 5 and $\frac{5}{1}$ *mates*. Also, 12 and $\frac{12}{1}$ are mates, 476 and $\frac{476}{1}$ are mates, and so on. In general, k and $\frac{k}{1}$ are mates. $U = N$.

3 and $\frac{3}{1}$ are mates, and 7 and $\frac{7}{1}$ are mates. The sums $3 + 7$ and $\frac{3}{1} + \frac{7}{1}$ are also mates. In general, $a + b$ and $\frac{a}{1} + \frac{b}{1}$ are mates. Therefore, N and

$\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \ldots\}$ are isomorphic with respect to addition.

Similarly $3 \cdot 7$ and $\frac{3}{1} \cdot \frac{7}{1}$ are mates, and, in general, ab and $\frac{a}{1} \cdot \frac{b}{1}$ are mates. Therefore, N and $\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \ldots\}$ are isomorphic with respect to multiplication. $\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \ldots\}$ is named set S in the lesson.

Because the members of N and set S behave in the same way with respect to addition and multiplication, we call the two sets "isomorphic."

The natural number 6 is not the same number as the rational number $\frac{6}{1}$. The natural number 6 is associated with the number of members in {0, 1, 2, 3, 4, 5} and each set equivalent to that standard set. The rational number $\frac{6}{1}$ is an infinite set of fractions, each of which is equivalent to the basic fraction $\frac{6}{1}$. In the mixed numeral $6\frac{1}{4}$, the symbol 6 really names the rational number $\frac{6}{1}$. Thus, $6\frac{1}{4}$ is really an abbreviated way of writing $\frac{6}{1} + \frac{1}{4}$. But, because of the isomorphism between sets N and S and the fact that 6 and $\frac{6}{1}$ are mates, the symbol 6 can be used to name $\frac{6}{1}$.

Notice that the rational number $\frac{12}{2}$ also is the mate of 6 because $\frac{12}{2}$ and $\frac{6}{1}$ indicate the same rational number. Thus, the mate of the natural number x is $\frac{xy}{y}$ when the universe for x is N and the universe for y is C.

At this point, we have developed the operations of multiplication and addition as used with two members of N or two members of R_a. We have not yet indicated how either operation can be used with two objects if one object is a member of N and one object is a member of R_a. Unless we define the multiplication of two objects chosen from different sets, the product of the natural number 10 and the rational number $\frac{1}{3}$ is not defined. The idea of the isomorphism between sets N and S gives meaning to the convention of expressing the product $\frac{10}{1} \cdot \frac{1}{3}$ by the numeral $10(\frac{1}{3})$. Because we know that 10 and $\frac{10}{1}$ are mates, we can think of $10(\frac{1}{3})$ as a member of R_a.

Unless the operation on two members of different sets is defined, the sum or the product of and $\frac{2}{3}$ cannot be rationalized without the idea of isomorphism. A precise explanation of a number named by a mixed numeral can involve the idea of isomorphism. The assertion that a rational number can be thought of as the quotient of a natural number and a counting number is meaningless unless an isomorphism between N and S is established first. Thus, $6 \div 5$ is meaningless in the set of natural numbers, but $\frac{6}{1} \div \frac{5}{1}$ is possible for the set of rational numbers of arithmetic. By using the idea of isomorphism and thinking of the mates $\frac{6}{1}$ and 6 and $\frac{5}{1}$ and 5, we can say that $6 \div 5 = \frac{6}{5}$ because $\frac{6}{1} \div \frac{5}{1} = \frac{6}{1} \times \frac{1}{5}$.

The words "isomorphic" and "isomorphism" are not used in the developmental exercises and probably should be used only with classes of able students. It should be explained that *isomorphism* means "same structure."

Instruction notes

■ Here is one possible way to introduce this lesson to a class. Write on the chalkboard: {0, 1, 2, 3, . . ., n, . . .}. Say: "We are going to map each member of the set of natural numbers onto a member of a subset of R_a." Under the tabulation of N, write: $\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \ldots\}$. Say: "We can map 0 onto $\frac{0}{1}$, 1 onto $\frac{1}{1}$, and so on. Onto what rational number will we map n?" [Onto $\frac{n}{1}$] "May we imagine that we can map each natural number onto a rational number that is indicated by a fraction that has the same natural number as its numerator?" [Yes] "What is the denominator of each fraction?" [One] "Is each of the rational numbers indicated by a basic fraction?" [Yes] Complete the picture on the board so that it looks like this:

$$\begin{array}{ccccccc} \{0, & 1, & 2, & 3, & \ldots, & n, & \ldots\} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ \{\frac{0}{1}, & \frac{1}{1}, & \frac{2}{1}, & \frac{3}{1}, & \ldots, & \frac{n}{1}, & \ldots\} \end{array}$$

Ask: "Are there any natural numbers between) and 1?" [No] "Are there any rational numbers between $\frac{0}{1}$ and $\frac{1}{1}$?" [Yes. Examples are $\frac{1}{4}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{7}{8}$.] "We will call the set $\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \ldots\}$ set S. Is S a subset of R_a?" [Yes] "Is set S equivalent to the set of natural numbers?" [Yes. The two sets are in one-to-one correspondence.] "Is set S an infinite set?" [Yes] "You will use sets N and S to learn an important mathematical idea. You have made use of this idea in your previous work in arithmetic.

Provision for individual differences

■ A class of abler students could do the work in this lesson as independent study. Be certain to review the main ideas with them after they have completed the developmental exercises.

You could tell your abler students that the relation set N has to set S is an isomorphic relation. Ask them if N and $\{\frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \ldots\}$ are isomorphic with respect to addition and multiplication. Note that $\{\frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \ldots\}$ is not set S. [They are isomorphic with respect to addition, but not with respect to multiplication. $a+b$ and $\frac{a+b}{2}$ are mates, but ab and $\frac{a}{2}(\frac{b}{2})$ are not mates. ab is the mate of $\frac{ab}{2}$.]

While slow learners may not grasp the basic idea of isomorphism, they will not have too much difficulty with the computation required in the exercises. Basically, it is important for them to understand that their previous work with examples such as $\frac{1}{3}(10)$ is consistent with what they learn in this book.

7 / 79 Learning computation / Pages 344-347

Mixed numerals used in computation

Objectives

■ The students get a better understanding of mixed numerals. They learn to convert mixed numerals to fraction numerals and fraction numerals to mixed numerals, and they use these numerals in computation.

Content overview

■ We introduced the idea of mates to get around the difficulties students experience when adding or multiplying numbers chosen from different sets. In this lesson the students use the idea of number mates in thinking of $3\frac{1}{5}$ as $\frac{3}{1}+\frac{1}{5}$.

In arithmetic, we converted the numeral $3\frac{1}{5}$ to $\frac{16}{5}$ by thinking as follows: $3\frac{1}{5}=3+\frac{1}{5}$. $1=\frac{5}{5}$ and $3=\frac{15}{5}$. $\frac{15}{5}+\frac{1}{5}=\frac{16}{5}$. Now we think of $3\frac{1}{5}$ as $\frac{3}{1}+\frac{1}{5}$ and use the sum property to obtain $\frac{3(5)+1(1)}{1(5)}$.

A mixed numeral is composed of a standard name of a counting number and a fraction numeral. Notice that a mixed numeral expresses a rational number.

Converting a mixed numeral to a fraction numeral may be accomplished by utilizing a familiar algorism. Using this "short cut," it is necessary first to find the product of the counting number and the denominator of the fraction. The sum of this product and the numerator is the numerator of the fraction that is to be obtained. The denominator of the fraction to be obtained is the denominator of the fraction that indicates the rational number expressed by the mixed numeral.

To convert a fraction numeral to a mixed numeral, first find the greatest multiple of the denominator that is less than the numerator. Unless the numerator is greater than the denominator, this multiple is zero, and there is no problem. If the numerator is greater than the denominator, think of the rational number as the sum of two rational numbers. Next find the natural-number mate of one of these two rational numbers used to form the sum. Then write a mixed numeral that expresses the rational number as a counting number and a rational number. For example, $\frac{29}{6}$ is converted by thinking of $\frac{24}{6}+\frac{5}{6}$, then $\frac{4}{1}+\frac{5}{6}$, then $4\frac{5}{6}$.

Computations such as those shown in D5 and D6 on page 346 were taught in earlier grades. Now we review this computation to obtain a better understanding of the processes involved. To perform this computation, we think of 5 as the rational number of arithmetic $\frac{5}{1}$; we think of $3\frac{1}{9}$ as the rational number of arithmetic $\frac{28}{9}$; and so on.

Instruction notes

■ You might begin this lesson by writing the numeral $2\frac{3}{4}$ on the chalkboard. Say: "Think about the number expressed by this numeral. Is it 2 plus $\frac{3}{4}$?" [Yes] "Have you learned how to obtain a sum from either two members of N or two members of R_a?" [Yes] "Have you learned to obtain a sum from a natural number and a rational number of arithmetic, that is, a member of N added to a member of R_a?" [No] "In this lesson you will learn what kind of number a mixed numeral names, and then you will learn some short cuts to use in computation."

Before discussing exercise G on page 345, try to have your students generalize for themselves the short cut for converting a mixed numeral to a fraction numeral.

After exercise D at the top of page 346, ask: "Is $\frac{7+5}{7}$ the sum of $\frac{7}{7}$ and $\frac{5}{7}$?" [Yes] "Why can we use the numeral $1\frac{5}{7}$ to name this number?" [$\frac{7}{7}$ is the same rational number as $\frac{1}{1}$, and the natural number 1 is the mate of the rational number $\frac{1}{1}$.]

After exercise W on page 346, write the numeral $\frac{17}{20}$ on the chalkboard. Ask: "Can we use a mixed numeral to name $\frac{17}{20}$?" [Yes] "Why don't we usually do it?" [There is no multiple of 20 that is less than 17 other than zero. The mixed numeral would be written $0\frac{17}{20}$.]

Provision for individual differences

■ The following developmental exercises could be omitted by a class of abler students:

H through L, page 345

F, G, I through N (column 1);

G through K (column 2), page 346

For a class of slow learners, you may choose to devote two periods to this lesson. On the first day, you might complete the lesson through exercise W on page 346 and assign "On your own" exercises 1 through 12 and 19 through 22. Finish the lesson on the second day and assign "On your own" exercises 13 through 18 and 23 through 38. Give students individual help, if possible.

If you devote two periods to this lesson for your average or ungrouped students, include the "Keeping skillful" exercises on page 347 in their assignment.

7 / 80 Exploring ideas / Pages 348-351

The commutative properties

Objectives

■ The students learn to use the transitive property of equality. They prove that multiplication of rational numbers of arithmetic has the commutative property, and they learn that there is a commutative property of addition for the rational numbers of arithmetic.

Content overview

■ Transitivity is a property of all equivalence relations, including the relation of equality. The relations of equivalence for fractions and equivalence for rate pairs have a transitive property. The transitive property of equality simply means that, when x, y, and z are replaced by rational numbers, and if $x = y \wedge y = z$, then $x = z$ for the same replacements.

The transitive property of equality can be expressed in two forms. Besides the form used on page 348, a second form is: if $x = y$ and $z = y$, then $x = z$. The second form can be used because of the symmetric property of equality, if $y = z$, then $z = y$.

Not all relations are transitive. For example, he social relation "is the natural son of" is not ransitive. If Charles is the natural son of Bob nd Bob is the natural son of Arthur, it does *not* ollow that Charles is the natural son of Arthur.

The transitive property of equality is used in a proof of the commutative property of multiplication of rational numbers of arithmetic. This proof s given below.

To prove: $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$. For a and c, U = N. For b and d, U = C.

1 $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.
Definition of product

2 $\frac{c}{d} \cdot \frac{a}{b} = \frac{ca}{db}$.
Definition of product

3 $\frac{ac}{bd} = \frac{ca}{db}$.
Commutative property of multiplication of natural numbers

4 $\frac{a}{b} \cdot \frac{c}{d} = \frac{ca}{db}$.
Transitive property of equality

5 $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$.
Transitive property of equality

Step 4 is derived by thinking that, if $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ and $\frac{ac}{bd} = \frac{ca}{db}$, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ca}{db}$. Step 5 is derived by thinking that, if $\frac{a}{b} \cdot \frac{c}{d} = \frac{ca}{db}$ and $\frac{c}{d} \cdot \frac{a}{b} = \frac{ca}{db}$, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$.

The following is a proof of the commutative property of addition of the rational numbers of arithmetic.

To prove: $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$. For a and c, U = N. For b and d, U = C.

1 $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$.
Sum property

2 $\frac{ad + bc}{bd} = \frac{da + cb}{db}$.
Commutative property of multiplication of natural numbers

3 $\frac{a}{b} + \frac{c}{d} = \frac{da + cb}{db}$.
Transitive property of equality

4 $\frac{da + cb}{db} = \frac{cb + da}{db}$.
Commutative property of addition of natural numbers

5 $\frac{a}{b} + \frac{c}{d} = \frac{cb + da}{db}$.
Transitive property of equality

6 $\frac{c}{d} + \frac{a}{b} = \frac{cb + da}{db}$.
Sum property

7 $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$.
Transitive property of equality

The commutative property of each operation with the rational numbers of arithmetic must be established, along with other properties, before we can view the rational numbers of arithmetic as a number system. The closure properties of the operations are developed in lessons 74 and 75; the associative properties in lesson 81; and the distributive property in lesson 82. Then, at the close of lesson 82, the rational numbers of arithmetic are seen to be a number system.

Instruction notes

■ Consider the following suggested class procedure. Say: "Turn back to page 272. Read exercise G. Have you used this idea several times this year?" [Yes, in congruence for segments on page 40, for example] "In this lesson, you will learn that this idea is an important property that is used over and over again in mathematics."

Before exercise C on page 348, write the following sentences on the chalkboard: $\frac{1}{2} = \frac{2}{4} \cdot \frac{2}{4} = \frac{10}{20}$. Ask: "Is each of these a true statement?" [Yes] "By using the transitive property of equality, what other true statement can you obtain?" [$\frac{1}{2} = \frac{10}{20}$.]

After exercise F on page 348, say: "In unit 6 you studied the commutative properties of addition and multiplication of natural numbers. What statements can you make about these properties?" [For each a and b, $a + b = b + a$ and $a \times b = b \times a$]

After the last paragraph on page 349, say: "You know that you used the idea of the transitive property of equality long before you began this lesson. Why do you have to learn now that

this idea is a property?" [It is used in the proof of the commutative property of multiplication of rational numbers.]

Provision for individual differences

■ For a class of abler students, the following developmental exercises could be omitted:

C through F, page 348
D through G, page 349
R, S, T, D, E, page 350

Encourage your abler students to find a proof for the commutative property of addition of rational numbers of arithmetic. One proof of this property is given in the "Content overview" for this lesson.

Have your abler students decide whether each of the following sentences expresses a true statement or a false statement:

1 If $a < b$ and $b < c$, then $a < c$. U = N. [True]

2 If $\overline{AB} \cong \overline{CD}$, and $\overline{CD} \cong \overline{KJ}$, then $\overline{AB} \cong \overline{KJ}$. [True]

3 If x is the successor of y, and y is the successor of z, then x is the successor of z. [False]

4 If Ralph is the father of Sam, and Sam is the father of Bill, then Ralph is the father of Bill. [False]

5 If Sally is older than Tom, and Tom is older than Sue, then Sally is older than Sue. [True]

6 If $x \neq y$ and $y \neq z$, then $x \neq z$. U = N. [False. x could be equal to z. $3 \neq 5$ and $5 \neq 3$, but $3 \neq 3$ is a false statement.]

For a class of slow learners, you can omit the part of the lesson that deals with a proof for the commutative property of multiplication in the system of rational numbers of arithmetic. This will reduce the teaching time needed for this lesson. To do this, omit display 4 and developmental exercises I through N on page 349. Use many examples like those in D3 and D5 along with the transitive property of equality to develop inductively the commutative properties of multiplication and addition of rational numbers.

The associative properties

Objectives

■ The students learn that the operations of addition and multiplication of rational numbers have the associative property, but that subtraction does not have this property.

Content overview

■ The set of rational numbers of arithmetic, together with two operations (multiplication and addition) and certain properties of the operations forms a number system. This system is developed in lesson 82. The associative properties of the two operations (addition and multiplication) are among the properties of the system.

The associative property of multiplication of rational numbers of arithmetic is expressed as $(xy)z = x(yz)$. The universe for each variable is R_a. The associative property of addition is expressed as $(x + y) + z = x + (y + z)$. The universe for each variable is R_a. Neither of these properties is proved in the student's book, but they are developed inductively.

A deductive proof of the associative property of multiplication of rational numbers of arithmetic is given below.

To prove: $(\frac{a}{b} \cdot \frac{c}{d})\frac{e}{f} = \frac{a}{b}(\frac{c}{d} \cdot \frac{e}{f})$. For a, c, and e, U = N. For b, d, and f, U = C.

1 $(\frac{a}{b} \cdot \frac{c}{d}) \cdot \frac{e}{f} = \frac{ac}{bd} \cdot \frac{e}{f}$.
Definition of product of two rational numbers

2 $\frac{ac}{bd} \cdot \frac{e}{f} = \frac{(ac)e}{(bd)f}$.
Definition of product of two rational numbers

3 $(\frac{a}{b} \cdot \frac{c}{d}) \cdot \frac{e}{f} = \frac{(ac)e}{(bd)f}$.
Transitive property of equality, (1) and (2)

4 $\frac{a}{b} \cdot (\frac{c}{d} \cdot \frac{e}{f}) = \frac{a}{b} \cdot \frac{ce}{df}$.
Definition of product of two rational numbers

5 $\frac{a}{b} \cdot \frac{ce}{df} = \frac{a(ce)}{b(df)}$.

Definition of product of two rational numbers

6 $\frac{a}{b} \cdot (\frac{c}{d} \cdot \frac{e}{f}) = \frac{a(ce)}{b(df)}$.

Transitive property of equality, (4) and (5)

7 $\frac{a(ce)}{b(df)} = \frac{(ac)e}{(bd)f}$.

Associative property of multiplication of natural numbers

8 $\frac{a}{b} \cdot (\frac{c}{d} \cdot \frac{e}{f}) = \frac{(ac)e}{(bd)f}$.

Transitive property of equality, (6) and (7)

9 $(\frac{a}{b} \cdot \frac{c}{d}) \cdot \frac{e}{f} = \frac{a}{b} \cdot (\frac{c}{d} \cdot \frac{e}{f})$.

Transitive property of equality, (3) and (8).

A deductive proof of the associative property of addition of rational numbers of arithmetic is given below.

To prove: $(\frac{a}{b} + \frac{c}{d}) + \frac{e}{f} = \frac{a}{b} + (\frac{c}{d} + \frac{e}{f})$. For a, c, and e, U = N. For b, d, and f, U = C.

1 $(\frac{a}{b} + \frac{c}{d}) + \frac{e}{f} = (\frac{ad + bc}{bd}) + \frac{e}{f}$.

Sum property of rational numbers

2 $(\frac{ad + bc}{bd}) + \frac{e}{f} = \frac{(ad + bc)f + (bd)e}{(bd)f}$

Sum property of rational numbers

3 $(\frac{a}{b} + \frac{c}{d}) + \frac{e}{f} = \frac{(ad + bc)f + (bd)e}{(bd)f}$.

Transitive property of equality, (1) and (2)

4 $\frac{(ad + bc)f + (bd)e}{(bd)f} = \frac{adf + bcf + bde}{bdf}$.

Distributive property, and commutative and associative properties of multiplication of natural numbers

5 $(\frac{a}{b} + \frac{c}{d}) + \frac{e}{f} = \frac{adf + bcf + bde}{bdf}$.

Transitive property of equality, (3) and (4)

6 $\frac{a}{b} + (\frac{c}{d} + \frac{e}{f}) = \frac{a}{b} + (\frac{cf + de}{df})$.

Sum property of rational numbers

7 $\frac{a}{b} + (\frac{cf + de}{df}) = \frac{a(df) + b(cf + de)}{bdf}$.

Sum property of rational numbers

8 $\frac{a}{b} + (\frac{c}{d} + \frac{e}{f}) = \frac{a(df) + b(cf + de)}{bdf}$.

Transitive property of equality, (6) and (7)

9 $\frac{a(df) + b(cf + de)}{bdf} = \frac{adf + bcf + bde}{bdf}$.

Distributive and associative properties for natural numbers

10 $\frac{a}{b} + (\frac{c}{d} + \frac{e}{f}) = \frac{adf + bcf + bde}{bdf}$.

Transitive property of equality, (8) and (9)

11 $(\frac{a}{b} + \frac{c}{d}) + \frac{e}{f} = \frac{a}{b} + (\frac{c}{d} + \frac{e}{f})$.

Transitive property of equality, (5) and (10)

Neither subtraction nor division of rational numbers has the associative property. This property is considered for division in lesson 85.

The associative property of addition may be used in computation involving addition of two rational numbers that are expressed by mixed numerals. D6 on page 354 shows how to obtain $(3 + 2) + (\frac{3}{4} + \frac{1}{8})$ from $3\frac{3}{4} + 2\frac{1}{8}$. Actually, this method does not have to be used for addition computation. Since $3\frac{3}{4} = \frac{15}{4}$ and $2\frac{1}{8} = \frac{17}{8}$, the sum $3\frac{3}{4} + 2\frac{1}{8}$ can be expressed as $\frac{15}{4} + \frac{17}{8}$ because of the well-defined property of addition of rational numbers. The sum property can be used to obtain $\frac{15(8) + 4(17)}{4(8)}$. However, as you see, the computation can become cumbersome when the associative property of addition is not used in computation of this sort.

Exercises F through J on page 354 describe how subtraction computation can be performed. Observe that $5\frac{2}{3}$ and $4\frac{20}{12}$ are the same rational number. The numeral $5\frac{2}{3}$ can be converted to the numeral $4\frac{20}{12}$, but the number is not "changed." To perform the subtraction computation, we can use the well-defined property and proceed as follows: $5\frac{2}{3} = 4\frac{20}{12}$ and $2\frac{3}{4} = 2\frac{9}{12}$. $4\frac{20}{12} - 2\frac{9}{12} = 5\frac{2}{3} - 2\frac{3}{4}$. $\frac{20}{12} - \frac{9}{12} = \frac{11}{12}$. $4 - 2 = 2$. $5\frac{2}{3} - 2\frac{3}{4} = 2\frac{11}{12}$.

Instruction notes

■ Before exercise A on page 352, ask: "What sentences express the associative properties of addition and multiplication of natural numbers?" [$(a + b) + c = a + (b + c)$ and $(ab)c + a(bc)$. U = N.]

After the first two paragraphs that follow exercise J on page 352, say: "Think of the sentence $(xy)z = x(yz)$. Can this sentence be used to express either the associative property of multiplication of rational numbers of arithmetic or the associative property of multiplication of natural numbers?" [Yes. If each replacement of the variables in the condition is made from N, the sen-

tence may be used to express the associative property of multiplication of natural numbers. If we use R_a as the universe, the sentence may be used to express the associative property of multiplication of rational numbers.] "Is the associative property of multiplication of natural numbers the same property as the associative property of rational numbers of arithmetic?" [No. One is a property of an operation on natural numbers, and the other is a property of an operation on rational numbers of arithmetic. For the variables in one condition, U = N, and for the variables in the other condition, $U = R_a$.] "Give another way to express the associative property of multiplication of rational numbers of arithmetic." $[(\frac{a}{b} \cdot \frac{c}{d}) \cdot \frac{e}{f} = \frac{a}{b} \cdot (\frac{c}{d} \cdot \frac{e}{f})$.]

Before the paragraph that precedes exercise J on page 353, ask: "What is another way in which this property can be expressed?" $[(\frac{a}{b} + \frac{c}{d}) + \frac{e}{f} = \frac{a}{b} + (\frac{c}{d} + \frac{e}{f})$.]

After exercise O on page 353, ask: "How many false statements do you need to obtain from a condition to know that the condition is not a property?" [Just one]

When discussing exercise I on page 354, write these sentences on the chalkboard:

$5\frac{8}{12} = 5 + \frac{8}{12}$. $5 + \frac{8}{12} = 4 + 1 + \frac{8}{12}$.

$4 + 1 + \frac{8}{12} = 4 + \frac{12}{12} + \frac{8}{12}$. $4 + \frac{12}{12} + \frac{8}{12} = 4 + \frac{20}{12}$.

$4 + \frac{20}{12} = 4\frac{20}{12}$.

Discuss briefly each of the statements expressed. Point out that, in this instance, the numeral 5 names the rational number $\frac{5}{1}$; the expression 4 + 1 names the sum $\frac{4}{1} + \frac{1}{1}$; and so on.

Provision for individual differences

■ For a class of abler students, you may omit the following developmental exercises:

A through E, and D1, page 352

A through E, and D3, page 353

You may want to show your abler students a proof of each of the associative properties of the operations on rational numbers of arithmetic. A proof of each is given in the "Content overview" for this lesson. An outstanding student might try to develop these (or other) proofs by himself.

For slow learners, show many applications of the use of each associative property. For example, have them simplify an expression such as $\frac{2}{5} + \frac{11}{12} + \frac{1}{12}$ in two ways. Point out that $\frac{2}{5} + (\frac{11}{12} + \frac{1}{12})$ is easily simplified, but that $(\frac{2}{5} + \frac{11}{12}) + \frac{1}{12}$ is not so easily simplified. As usual, give these students as much individual help as you can.

---

7 / 82 Exploring ideas / Pages 355-359

The distributive property and the system of rational numbers

Objectives

■ The students learn that the operation of multiplication distributes over addition of rational numbers of arithmetic. Then they learn that the set of rational numbers of arithmetic, together with two operations and their properties, forms a number system.

Content overview

■ The distributive property is developed inductively in the student's book.

Remember that the distributive property relates two operations. The operation # distributes over the operation ★ only if the condition $a \# (b \bigstar c) = (a \# b) \bigstar (a \# c)$ is satisfied for all a, b, and c.

A deductive proof of the distributive property of multiplication over addition of rational numbers of arithmetic is given below.

To prove: $\frac{a}{b} \cdot (\frac{c}{d} + \frac{e}{f}) = \frac{a}{b}(\frac{c}{d}) + \frac{a}{b}(\frac{e}{f})$. For a, c, and e, U = N. For b, d, and f, U = C.

1 $\frac{a}{b} \cdot (\frac{c}{d} + \frac{e}{f}) = \frac{a}{b} \cdot (\frac{cf + de}{df})$.

Sum property of two rational numbers of arithmetic

2 $\frac{a}{b} \cdot (\frac{cf + de}{df}) = \frac{a(cf + de)}{b(df)}$.
Definition of product of two rational numbers of arithmetic
3 $\frac{a}{b} \cdot (\frac{c}{d} + \frac{e}{f}) = \frac{a(cf + de)}{b(df)}$.
Transitive property of equality, (1) and (2)
4 $\frac{a(cf + de)}{b(df)} = \frac{acf + ade}{bdf}$.
Distributive property of natural numbers and associative property of multiplication of natural numbers
5 $\frac{a}{b} \cdot (\frac{c}{d} + \frac{e}{f}) = \frac{acf + ade}{bdf}$.
Transitive property of equality, (3) and (4)
6 $\frac{a}{b}(\frac{c}{d}) + \frac{a}{b}(\frac{e}{f}) = \frac{ac}{bd} + \frac{ae}{bf}$.
Definition of product of two rational numbers of arithmetic
7 $\frac{ac}{bd} + \frac{ae}{bf} = \frac{(ac)(bf) + (bd)(ae)}{(bd)(bf)}$.
Sum property of two rational numbers of arithmetic
8 $\frac{a}{b}(\frac{c}{d}) + \frac{a}{b}(\frac{e}{f}) = \frac{(ac)(bf) + (bd)(ae)}{(bd)(bf)}$.
Transitive property of equality, (6) and (7)
9 $\frac{(ac)(bf) + (bd)(ae)}{(bd)(bf)} = \frac{b(acf + dae)}{b(bdf)}$.
Distributive property and commutative and associative properties of multiplication of natural numbers
10 $\frac{a}{b}(\frac{c}{d}) + \frac{a}{b}(\frac{e}{f}) = \frac{b(acf + dae)}{b(bdf)}$.
Transitive property of equality, (8) and (9)
11 $\frac{b(acf + dae)}{b(bdf)} = \frac{acf + dae}{bdf}$.
Reduction property of rational numbers
12 $\frac{a}{b}(\frac{c}{d}) + \frac{a}{b}(\frac{e}{f}) = \frac{acf + dae}{bdf}$.
Transitive property of equality, (10) and (11)
13 $\frac{acf + dae}{bdf} = \frac{acf + ade}{bdf}$.
Commutative and associative properties of multiplication of natural numbers
14 $\frac{a}{b}(\frac{c}{d}) + \frac{a}{b}(\frac{e}{f}) = \frac{acf + ade}{bdf}$.
Transitive property of equality, (12) and (13)
15 Therefore, $\frac{a}{b} \cdot (\frac{c}{d} + \frac{e}{f}) = \frac{a}{b}(\frac{c}{d}) + \frac{a}{b}(\frac{e}{f})$.
Transitive property of equality, (5) and (14)

The above is a proof of the "left-hand" distributive property. A proof of the "right-hand" distributive property is given below.

To prove: $(y + z)x = yx + zx$. For each variable, $U = R_a$.

1 $(y + z)x = x(y + z)$.
Commutative property of multiplication
2 $x(y + z) = xy + xz$.
Distributive property of multiplication over addition ("left-hand")
3 $(y + z)x = xy + xz$.
Transitive property of equality, (1) and (2)
4 $xy + xz = yx + zx$.
Commutative property of multiplication
5 $(y + z)x = yx + zx$.
Transitive property of equality, (3) and (4)

The condition $(y + z)x = yx + zx$ is the "right-hand" distributive property of multiplication over addition of rational numbers of arithmetic.

The operation of addition does not distribute over the operation of multiplication because a true statement is not always obtained from $x + (yz) = (x + y)(x + z)$. $U = R_a$. For example, $3 + (2 \cdot 4) \neq (3 + 2)(3 + 4)$.

Multiplication distributes over subtraction; that is, $x(y - z) = xy - xz$ provided $y - z$ is a rational number of arithmetic. $y - z$ is a member of R_a only when the replacement for y is equal to or greater than the replacement for z. If $y \geqq z$ is satisfied, then $x(y - z) = xy - xz$ is a property. For each variable, $U = R_a$.

The distributive property is developed for two reasons: it is one of the properties of a number system, and it gives meaning to certain rules in computation involving members of R_a. We wish to show that R_a, together with two operations and certain properties, is a number system and to perform calculations within this system.

The mixed numeral for the expression $5(2\frac{3}{4})$ can be found by using the distributive property. $5(2\frac{3}{4})$ means $\frac{5}{1}(\frac{2}{1} + \frac{3}{4})$. $\frac{5}{1}(\frac{2}{1} + \frac{3}{4}) = \frac{5}{1}(\frac{2}{1}) + \frac{5}{1}(\frac{3}{4})$. $\frac{5}{1}(\frac{2}{1}) + \frac{5}{1}(\frac{3}{4}) = \frac{10}{1} + \frac{15}{4}$. $\frac{10}{1} + \frac{15}{4} = \frac{10}{1} + \frac{3}{1} + \frac{3}{4}$. $\frac{10}{1} + \frac{3}{1} + \frac{3}{4} = \frac{13}{1} + \frac{3}{4}$. $\frac{13}{1} + \frac{3}{4} = 13\frac{3}{4}$.

The distributive property is the last property that we need to identify before we can consider

the system of R_a. The set of rational numbers of arithmetic, the two operations (addition and multiplication), and the closure, commutative, and associative properties of each operation were developed in this unit in prior lessons. R_a, together with the operations and properties named in D11 on page 358, forms the number system called the system of rational numbers of arithmetic. Notice that this number system is developed without considering either division or the identity-element properties. The identity elements are considered in lesson 83, and division is developed in lesson 85.

Instruction notes

■ All but the most able students should devote two periods to this lesson. On the first day, complete the developmental exercises through W on page 357 and assign "On your own" exercises 1, 2, and 4 through 23. On the second day, finish the lesson and assign "On your own" exercises 3 and 24 through 35.

Before beginning the developmental exercises, review with your students the technique involved in simplifying an expression. You may wish to refer them to lesson 65, pages 279 through 281.

After exercise S on page 357, point out to your students that the transitive property of equality can be used for the statements expressed in D8 to obtain $25(104\frac{1}{3}) = 2608\frac{1}{3}$.

After exercise F in column 2 on page 358, say: "Compare D1 on page 282 and D11 on page 358. Notice that each display names a set, two operations, and some properties of the operations. Any set of objects that has two operations and these properties of the operations is a number system. You will learn about other number systems in your future study of mathematics."

Provision for individual differences

■ For a class of very able students, complete this lesson in one period by omitting the following developmental exercises:

B, C, D, page 355
H, I, J, L, M, page 356
T through W, page 357
C, D, page 358

You may want to show these students a proof of the distributive property of multiplication over addition of rational numbers. A proof of this property is given in the "Content overview" for this lesson.

Abler students might also consider the following exercises:

1 $S = \{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \ldots\}$. Is set S a subset of R_a? [Yes]

2 Does set S form a number system with respect to addition and multiplication? [Yes]

3 Does set S form a number system with respect to addition and subtraction? [No. Set S is not closed under subtraction. For example, there is no member of S that satisfies $\frac{1}{1} - \frac{2}{1} = x$.]

4 Tabulate the set of rational numbers of arithmetic that are mates of the even natural numbers. $[\{\frac{0}{1}, \frac{2}{1}, \frac{4}{1}, \ldots\}]$

5 Is the set you tabulated for exercise 4 a subset of R_a? [Yes] Name this set "T."

6 Does set T form a number system with respect to addition and multiplication? [Yes]

7 Does every subset of R_a form a number system with respect to addition and multiplication? [No. For example, most finite subsets of R_a are not closed under either operation. The set $\{\frac{1}{1}, \frac{3}{1}, \frac{5}{1}, \ldots\}$ is not closed under addition.]

If you want to reduce the teaching time needed for this lesson, or if you have a class of slow learners, omit exercise O on page 356, exercises A through D on page 357, display 12, and developmental exercises F through J on pages 358 and 359.

Your slow learners should accept the rational numbers of arithmetic as a number system. However, it is not essential that they know which properties are required for a number system.

7 / 83 Exploring ideas / Pages 360-362

The identity elements

Objectives

■ The students learn that the rational number 0 is the identity element for addition of rational numbers; the rational number 1 is the identity element for multiplication of rational numbers; and the product of the rational number 0 and any other rational number is 0.

Content overview

■ In this lesson we speak of the sum of a rational number and 0. What we mean is the sum of a rational number of arithmetic and the rational-number mate of the natural number 0. Also, when we speak of the product of a rational number and 1, we mean the product of a rational number of arithmetic and the rational-number mate of the natural number 1. We use this language in the student's book because it facilitates communication of ideas. The meaning of this language has been established in previous lessons.

The identity element for addition of rational numbers of arithmetic is the number $\frac{a}{b}$ that yields a true statement from $\frac{a}{b} + \frac{c}{d} = \frac{c}{d}$ for each replacement of c and d. Many rational numbers could be tested as replacements for $\frac{a}{b}$, but since it is known that 0 is the identity element for addition of natural numbers, it makes sense to test first the rational-number mate of 0. In exercises C through H on page 360, we show that $\frac{0}{b} + \frac{c}{d} = \frac{c}{d}$. The rational number 0 is the identity element for addition of rational numbers, and the identity-element property of addition of rational numbers is expressed as $0 + x = x$. $U = R_a$.

The identity element for multiplication of rational numbers is the number $\frac{a}{b}$ that yields a true statement from $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d}$ for each replacement of c and d. From the definition of product of two rational numbers, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. But if $\frac{a}{b}$ is a rational number and $a = b$, then $\frac{ac}{bd} = \frac{c}{d}$ because of the reduction property of fractions. The rational number 1 is indicated by the fraction $\frac{a}{b}$ when $a = b$ is satisfied. Thus, the rational number 1 is the identity element for multiplication of rational numbers, and the identity-element property of multiplication of rational numbers is expressed as $1 \cdot x = x$. $U = R_a$.

Other properties can be derived from the identity-element property of addition and the relation of subtraction to addition. These two ideas can be used to obtain the properties $x - x = 0$ and $x - 0 = x$.

Because of the commutative property of multiplication, the property $x \cdot 1 = x$ can be obtained from $1 \cdot x = x$. Other properties that are derived from the relation of division to multiplication and the identity-element property of multiplication cannot be considered until the idea of division of rational numbers is established. Division of rational numbers is developed in lesson 85.

The zero property of multiplication of rational numbers is expressed as $0 \cdot x = 0$. $U = R_a$. The numeral 0, of course, names the rational number 0.

Instruction notes

■ In lesson 68 the students learned that 0 is the identity element for addition of natural numbers and that 1 is the identity element for multiplication of natural numbers. To develop the identity element for addition of rational numbers, you may want to use the following procedure instead of developmental exercises C through H on page 360.

Before exercise C on page 360, write on the chalkboard the following: $\frac{0}{1} + \frac{c}{d} = \frac{0 \cdot d + 1 \cdot c}{1 \cdot d}$. The universe for c is N and for d is C. Ask: "Does the

symbol $\frac{0}{1}$ express the rational number 0?" [Yes] "What is the product of 0 and d?" [0] "What is the product of 1 and c?" [c] "What is the product of 1 and d?" [d] Write on the board: $\frac{0 \cdot d + 1 \cdot c}{1 \cdot d} = \frac{0+c}{d}$. Ask: Does this sentence express a true statement for each c and d?" [Yes] Write: $\frac{0}{1} + \frac{c}{d} = \frac{0+c}{d}$. Ask: "What property do we use to obtain this condition?" [Transitive property of equality] "What is the sum of 0 and c?" [c] "Is $\frac{0+c}{d}$ equal to $\frac{c}{d}$?" [Yes] "Can we say that $\frac{0}{1} + \frac{c}{d} = \frac{c}{d}$?" [Yes] "Do we always obtain a true statement from this condition?" [Yes]

After exercise G on page 361, write on the chalkboard: $\frac{a}{a} \times \frac{c}{d} = \frac{ac}{ad}$. Ask: "Is the rational number 1 indicated by a fraction whose numerator is the same counting number as the denominator?" [Yes] "How do you know that the fractions $\frac{c}{d}$ and $\frac{ac}{ad}$ indicate the same rational number?" [They are equivalent fractions.]

Provision for individual differences

■ The abler students could give a solution of each of the following conditions. $U = R_a$.

1 $\frac{12}{12} + x = \frac{7}{7}$. [0]
2 $\frac{7}{7}(x) = \frac{12}{12}$. [1]
3 $\frac{4}{9}(2\frac{1}{2} - x) = 0$. [$2\frac{1}{2}$]
4 $x(\frac{0}{7}) = \frac{0}{9}$. [Any member of R_a]
5 $x(x+1) = 0$. [0]

For slow learners, emphasize the use of each property through extra examples such as $0 + \frac{2}{3} = \frac{2}{3}$ and $\frac{9}{13} \times 1 = \frac{9}{13}$.

This lesson may be omitted. The idea of identity elements was developed in lesson 68.

---

7 / 84 Exploring ideas / Pages 363-365

Reciprocals and the reciprocal property

Objectives

■ The students learn the definition of the term reciprocal and study the reciprocal property of rational numbers of arithmetic.

Content overview

■ If the product of two rational numbers is 1, then each of the two rational numbers is the *reciprocal* of the other. Every non-zero rational number has a reciprocal.

The definition of reciprocal and the reciprocal property are used in computation and also in proofs of other properties, including the quotient property of rational numbers, which is developed in lesson 85.

The fraction $\frac{5}{6}$ indicates the rational number $\frac{5}{6}$. To obtain a fraction that indicates the reciprocal of the number $\frac{5}{6}$, the components of the fraction $\frac{5}{6}$ can be interchanged to obtain the fraction $\frac{6}{5}$. The fraction $\frac{6}{5}$ indicates the rational number $\frac{6}{5}$, which is the reciprocal of the rational number $\frac{5}{6}$. Reciprocals are rational numbers, not fractions.

A numerator and a denominator of a fraction cannot actually be interchanged. When we say "interchange," we mean that you should think of the rational number that is indicated by a fraction whose numerator is the denominator of the given fraction and whose denominator is the numerator of the given fraction.

The *reciprocal property* can be proved by the following steps.

To prove: $\frac{a}{b} \cdot \frac{b}{a} = 1$. $U = C$.

1 $\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba}$.
Definition of the product of two rational numbers of arithmetic

2 $\frac{ab}{ba} = \frac{ab}{ab}$.
Commutative property of multiplication of natural numbers

3 $\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ab}$.
Transitive property of equality

4 $\frac{ab}{ab} = \frac{1}{1}$.
Reduction property of rational numbers of arithmetic and the definition of a rational number of arithmetic

5 $\frac{a}{b} \cdot \frac{b}{a} = \frac{1}{1}$.
Transitive property of equality

Because the product of $\frac{a}{b}$ and $\frac{b}{a}$ is the rational number 1, $\frac{a}{b}$ and $\frac{b}{a}$ are reciprocals of each other by the definition of reciprocal.

A non-zero rational number has exactly one reciprocal. The reciprocal of $\frac{2}{5}$ is $\frac{5}{2}$. However, the rational number $\frac{5}{2}$ may also be indicated by the fractions $\frac{10}{4}$, $\frac{15}{6}$, $\frac{20}{8}$, and so on. A reciprocal is a number. In general, the reciprocal of the rational number $\frac{a}{b}$ is the rational number indicated either by the fraction $\frac{b}{a}$ or some fraction equivalent to $\frac{b}{a}$. The universe for each variable is C.

The rational number 0 does not have a reciprocal because the product of 0 and any rational number is 0; it is never 1.

Instruction notes

■ After exercise A on page 363, ask: "What does each of the pairs used to form a product have in common?" [The product is 1.] "What are some other pairs of rational numbers whose product is 1?" [$\frac{1}{3}$ and 3, $\frac{4}{5}$ and $\frac{5}{4}$, $\frac{1000}{7}$ and $\frac{7}{1000}$, etc.]

Before exercise O on page 363, say: "Remember that the numeral $2\frac{2}{3}$ in D2 expresses the rational number $\frac{2}{1}+\frac{2}{3}$, or $\frac{8}{3}$."

After exercise C on page 364, ask: "What is the product of $\frac{1}{4}$ and $\frac{20}{5}$?" [$\frac{20}{20}$, or 1] "You know that $\frac{5}{20}$ and $\frac{1}{4}$ can each be used as the reciprocal of $\frac{20}{5}$. Does a number have more than one reciprocal?" [No. $\frac{1}{4}$ and $\frac{5}{20}$ are the same rational number.]

After exercise K on page 364, say: "Think again about the reciprocal property. Can you say that the reciprocal of $\frac{a}{b}$ is $\frac{kb}{ka}$?" [Yes, when the universe for each variable is C] Prove this assertion for your students. [$\frac{a}{b}\cdot\frac{kb}{ka}=\frac{akb}{bka}$. $\frac{akb}{bka}=\frac{a(kb)}{(bk)a}$. $\frac{a(kb)}{(bk)a}=\frac{a(kb)}{a(bk)}\cdot\frac{a(kb)}{a(kb)}=\frac{1}{1}$. $\frac{a}{b}\cdot\frac{kb}{ka}=\frac{1}{1}$. To obtain the last condition, use the transitive property of equality.]

Provision for individual differences

■ Have abler students consider the following exercises:

1 (a) What is the sum of $1\frac{4}{5}$ and $3\frac{3}{5}$? [$5\frac{2}{5}$] What is the reciprocal of this sum? [$\frac{5}{27}$]

(b) What is the reciprocal of $1\frac{4}{5}$? [$\frac{5}{9}$] Of $3\frac{3}{5}$? [$\frac{5}{18}$] What is the sum of the two reciprocals? [$\frac{15}{18}$]

(c) Is the reciprocal of the sum of any two numbers equal to the sum of the reciprocals of the two numbers? [No]

2 (a) What is the product of $\frac{5}{2}$ and $\frac{1}{3}$? [$\frac{5}{6}$] What is the reciprocal of this product? [$\frac{6}{5}$]

(b) What is the reciprocal of $\frac{5}{2}$? [$\frac{2}{5}$] Of $\frac{1}{3}$? [$\frac{3}{1}$] What is the product of the two reciprocals? [$\frac{6}{5}$]

(c) Is the reciprocal of the product of any two numbers equal to the product of the reciprocals of the two numbers? [Yes]

3 If x is the reciprocal of y and y is the reciprocal of z, how are x and z related? [They are equal.]

4 If $xyz = 1$ and y and z are reciprocals of each other, then what number must x be to obtain a true statement? [The rational number 1]

Give slow learners extra practice in finding the reciprocals of rational numbers.

---

7 / 85 Exploring ideas / Pages 365-369

Division of rational numbers

Objectives

■ The students learn the meaning of division of rational numbers and the meaning of the quotient property of rational numbers of arithmetic. They discover that the set of non-zero rational numbers is closed under division, and they learn to express the reciprocal of a non-zero rational number x as $1/x$. They also learn a new way of expressing the quotient of two rational numbers.

Content overview

■ In *STM 1*, division is given meaning in terms of its relation to multiplication. We say that $\frac{a}{b}\div\frac{c}{d}=y$ means $\frac{a}{b}=y\cdot\frac{c}{d}$.

Later, in *STM 2*, division of all rational numbers (including negative rational numbers) is given meaning in terms of a multiplicative inverse; that is, $x \div y$ means $x(\frac{1}{y})$.

The quotient property of rational numbers of arithmetic is developed inductively in this lesson. The property is expressed as follows: The universe for a is N. The universe for b, c, and d is C. For each replacement of a, b, c, and d, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

The other properties developed in this lesson apply only to the set of non-zero rational numbers of arithmetic. The students will use this set as the universe for variables in many instances in the future. On page 396, in lesson 90, the letter Z is introduced to name the set of non-zero rational numbers of arithmetic.

The set Z is closed under division. When the universe for each variable is C, a true statement is always obtained from $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$. Since $\frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$ by the definition of the product of two rational numbers, and since ad and bc are counting numbers, $\frac{ad}{bc}$ is a non-zero rational number. Because the product $\frac{a}{b} \cdot \frac{d}{c}$ is a rational number, the quotient $\frac{a}{b} \div \frac{d}{c}$ must also be a rational number.

Remember that multiplication of rational numbers of arithmetic has a well-defined property and that division is related to multiplication. The numbers used to form a quotient may be replaced by equal numbers to obtain the same quotient.

The condition $\frac{a}{b} \div 1 = \frac{a}{b}$ means $\frac{a}{b} = \frac{a}{b} \cdot 1$. $\frac{a}{b} \div 1 = \frac{a}{b}$ is a property because we get true statements for all replacements of the variables.

The quotient of 0 and any non-zero rational number is 0. Since $\frac{0}{a} \div \frac{b}{c} = \frac{0}{a} \times \frac{c}{b}$ and $\frac{0}{a} \times \frac{c}{b} = 0$, $\frac{0}{a} \div \frac{b}{c} = 0$.

Division of rational numbers has neither an associative property nor a commutative property.

The idea of the isomorphism between N and $\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \ldots\}$ is used again on page 368. The quotient property of rational numbers can be used to obtain $\frac{a}{b}$ from $\frac{a}{1} \div \frac{b}{1}$. Because a and b are the natural-number mates of $\frac{a}{1}$ and $\frac{b}{1}$, $a \div b$ can be thought of as $\frac{a}{b}$. This idea is very important in mathematics. For one thing, it allows us to convert fraction numerals to decimal numerals. For example, the decimal numeral that names $\frac{3}{4}$ can be found by dividing 3 by 4. While there is no natural-number quotient $3 \div 4$, we can use $\frac{3}{1} \div \frac{4}{1}$ to obtain a quotient that is a rational number.

Two new ways of indicating division are introduced in exercise R on page 368. Note that these symbols as used here do not name fractions or rate pairs. Therefore, either symbol expresses "divided by." The reciprocal and quotient properties are restated using this new symbolism.

Instruction notes

■ It is expected that most classes will need to devote two periods to this lesson. On the first day, complete through exercise Y on page 367 and assign "On your own" exercises 1 through 15. On the second day, you should spend a few minutes on review. Then finish the lesson and assign "On your own" exercises 16 through 34.

For each sentence in D2 and D3, be certain that your students understand how each condition or statement expressed is derived before considering the idea expressed by the next sentence.

After exercise A on page 367, ask: "How do you know if a numeral expresses the rational number 0?" [The numerator of the fraction that indicates the rational number is always expressed by the numeral 0.]

After exercise P on page 368, say: "Is there a natural number that is the quotient of 49 and 20?" [No] Write "$49 \div 20$" on the chalkboard. Say: "If the numeral 49 and the numeral 20 name natural numbers, does the expression on the board name a natural number?" [No. Division of natural numbers is not closed.] "You know that you can think of the rational-number mates of an

ordered pair of natural numbers as having a quotient. You can use this idea in converting certain fraction numerals to mixed numerals, and later you will find how to use this idea to convert fraction numerals to decimal numerals."

Provision for individual differences

■ A class of abler students may omit the following developmental exercises:

I, K, page 366
V through Y, E, F, page 367
L through P, page 368

Mastery of the content of this lesson should be a challenge for your students. Your abler students should try to develop the proof for exercise N on page 367. A proof is shown below.

To prove: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$. For a, U = N. For b, c, and d, U = C.

1 $\frac{a}{b} \div \frac{c}{d} = y$.

Assumption that the quotient of a member of R_a and a member of Z is a rational number

2 $\frac{a}{b} = y \cdot \frac{c}{d}$.

Relationship of multiplication and division

3 $\frac{a}{b} \cdot \frac{d}{c} = (y \cdot \frac{c}{d})\frac{d}{c}$.

Well-defined property of multiplication of rational numbers

4 $\frac{a}{b} \cdot \frac{d}{c} = y(\frac{c}{d} \cdot \frac{d}{c})$.

Associative property of multiplication of rational numbers

5 $\frac{a}{b} \cdot \frac{d}{c} = y(1)$.

Reciprocal property of rational numbers

6 $\frac{a}{b} \cdot \frac{d}{c} = y$.

Identity-element property of multiplication of rational numbers.

7 Therefore, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$.

Transitive property of equality (1) and (6)

Expect slow learners to have some difficulty with the derivation of the quotient property of rational numbers. Be satisfied if they understand that the quotient property has a mathematical basis. Emphasize the manipulative aspects of division computation involving rational numbers.

7 / 86 Exploring ideas / Pages 370-374

Decimal numerals

Objectives

■ The students learn that decimal numerals are names for rational numbers and that they are a part of the decimal numeration system presented in unit 5.

Content overview

■ Lesson 86 is the first of three lessons about decimal fraction numerals.

A decimal numeral is a numeral that is written in a base-ten system of numeration. Other kinds of numerals are: tally, code, grouping, fraction numerals, and so on. A number is classified according to the set of which it is a member. Some sets of numbers are: natural, rational numbers of arithmetic, rational, and real. (Rational numbers and real numbers are studied in *STM 2*.) Since we make a distinction between a number and a numeral, it is necessary for us to think of a "decimal" as a numeral.

Since decimal numerals are symbols, they are used to name numbers. In unit 5, the decimal system of numeration is used to name natural numbers. In unit 7, the decimal system of numeration is used to name rational numbers of arithmetic.

Remember that the decimal system of numeration has the properties of base, place value, repetition, and addition.

The rational numbers of arithmetic considered in this lesson are those that can be indicated by fractions whose denominators are powers of 10. The numeral 24.73 expresses the number $\frac{24}{1} + \frac{73}{100}$, or $\frac{2473}{100}$. The expanded form of the numeral 24.73 is $2(10^1) + 4(1) + 7(\frac{1}{10^1}) + 3(\frac{1}{10^2})$. Note that the nu-

meral $10^0 = 1$, used as another name for 1, is introduced in *STM 2*. Therefore, the numeral 1 is used in *STM 1* to name the number associated with the 10^0 position. Also, the use of negative integers as exponents is presented in *STM 2*. In *STM 1*, the number associated with each position to the right of the decimal point is presented as a reciprocal of a power of 10.

Because we think of a point as an idea and a dot as a symbol, the name "decimal dot" would have been a more logical choice than "decimal point" as a name for the dot in a numeral. However, since the expression "decimal point" is commonly used, we have yielded to convention on this matter.

Decimal fraction numerals are often easier to use in computation than common fraction numerals because finding a common denominator is taken care of "automatically" by the algorism.

Computing with decimals is developed in lesson 87. Converting common fraction numerals to decimal fraction numerals is developed in lesson 88.

Instruction notes

■ Begin the lesson by reviewing the properties of the decimal system of numeration: base, place value, addition, and repetition. Write the numeral 333 on the chalkboard and have the students tell how the properties are exemplified in that numeral. [The digit 3 is used more than once; base and place value determine the number associated with each position in which there is a digit; and the additive property is demonstrated by the fact that 333 means $3(10^2) + 3(10^1) + 3(1)$.]

Before exercise D on page 370, ask: "Is $972\frac{564}{1000}$ a natural number?" [No] "Is it a rational number?" [Yes] "In unit 5 you learned how to write the expanded form of a numeral for a natural number. Now you will learn how to write the expanded form of a numeral for a rational number of arithmetic."

After exercise F on page 370, say: "Look at numeral D in D2. How would you read this numeral as it is expressed in the display?" ["5 over 10 to the first power plus 6 over 10 squared plus 4 over 10 cubed"]

Before the paragraph that follows exercise O on page 371, ask: "Could you also say that the number associated with each digit is ten times the number that is associated with the digit to its right?" [Yes]

After exercise Z on page 372, write the numerals .802 and 800.002 on the chalkboard. Ask: "How do you read each of these numerals?" [802 thousandths; 800 and 2 thousandths] Say: "As you can see, it is important to use the word 'and' properly in reading a numeral."

Most classes should devote two periods to this lesson. On the first day, complete through exercise Z on page 372 and assign "On your own" exercises 1 through 7 and exercises 13, 14, 17, 20, 21, 27, 28, and 30. On the second day, finish the lesson and assign the remaining "On your own" exercises.

Provision for individual differences

■ For a class of abler students, you could use this lesson as a self-study assignment, or you could omit the following developmental exercises in a class discussion:

B, E, page 370
L, M, N, R, page 371
U through X, D through G, page 372
O, U, V, B, C, page 373
F, J through N, page 374

Average and abler students could fill in the missing symbols in a chart such as the one shown on the next page. Have copies made of the chart, if possible.

Have slow learners work with an abacus and with place-value charts. If your abacus is made of wood or has a wooden frame, use a thumbtack to represent the decimal point.

Decimal numeral	Expanded form	Fraction numeral or mixed numeral
.63	$6(\frac{1}{10^1}) + 3(\frac{1}{10^2})$	$\frac{63}{100}$
.755		
	$4(\frac{1}{10^1}) + 0(\frac{1}{10^2}) + 2(\frac{1}{10^3})$	
		$\frac{7}{10000}$
1.2254		
		$28\frac{7}{10}$
	$3(10^1) + 9(1) + 5(\frac{1}{10^1}) + 1(\frac{1}{10^2})$	
8.0079		

7 / 87 Learning computation / Pages 375-382

Computing with decimals

Objectives

■ The students learn to compute with decimal numerals. They also learn to use decimals to find reciprocals of rational numbers of arithmetic.

Content overview

■ Computation with decimals is often quicker and easier than computation with mixed numerals. For example, the difference $3\frac{7}{40} - 2\frac{9}{25}$ probably can be found more easily when it is expressed as $3.175 - 2.360$. Also, for practical measurement problems, measures expressed as decimals are easier to work with. This is particularly true if the measures are expressed in metric units. Thus, there is a practical reason for attaining skill in computation with decimals.

Addition and subtraction involving numbers expressed as decimals can be confusing if the numerals are incorrectly arranged like this:

$$\begin{array}{r} 3.8\,4 \\ \underline{5.7} \end{array}$$

The possibility of this situation occurring is greatly diminished if we emphasize the use of the sum property of rational numbers as applied to numbers expressed as decimals. Understanding of this property makes it clear that in this situation 5.7 should be expressed as 5.70.

Notice how the definition of the sum of two rational numbers is applied in D1 on page 375 and how the difference property is applied in D2 on the same page. Computation with decimals is given meaning by using mixed numerals and the properties of the decimal system of numeration.

Multiplication involving decimals is performed like multiplication involving numerals for natural numbers, with the exception that a decimal point is written in the numeral that expresses the product. The rule for locating the decimal point in this numeral is determined by the product of the denominators of the fractions that indicate the rational numbers being multiplied. For example, consider the product $.73 \times .05$. The numerals .73 and .05 express the rational numbers $\frac{73}{100}$ and $\frac{5}{100}$. By the definition of the product of two rational numbers, $\frac{73}{100} \cdot \frac{5}{100} = \frac{73(5)}{100(100)}$. Since $100 \cdot 100 = 10{,}000$, the product is a rational number indicated by a fraction whose denominator is 10,000. Therefore, the decimal that expresses the product will be in ten-thousandths and $.73 \times .05 = .0365$. In general, the number of digits to the right of the decimal point in a decimal that expresses a product is the sum of the number of digits to the right of the decimal points in the numerals that express the rational numbers being multiplied.

Notice that the "times sign" ($\times$) is normally used rather than the raised dot to indicate multiplication that involves decimals. This procedure prevents any confusion that might arise if the multiplication dot and the decimal point should both occur in an expression.

The product of a given number and a power of 10 can be found quickly by changing the posi-

tion of each digit in the numeral that expresses the given number; that is, $7.092 \times 10^1 = 70.92$, $7.092 \times 10^2 = 709.2$, $7.092 \times 10^3 = 7092$, and $7.092 \times 10^4 = 70920$. Notice that the number of places in the relocation of each digit is the same as the exponent that is used to form the product. Because "changing the position of each digit in the numeral" is actually an ambiguous phrase, we refer in *STM 1* to the number of digits as "to the left of the decimal point."

From our work in lesson 85, we know that $\frac{3}{1} \div \frac{4}{1} = \frac{3}{4}$ and that $\frac{3}{4} = \frac{3}{4} \cdot \frac{10}{10}$ by the identity-element property of multiplication of rational numbers. $\frac{3}{4} \cdot \frac{10}{10} = \frac{3 \times 10}{4 \times 10}$ by the definition of product of two rational numbers. $\frac{3}{1} \div \frac{4}{1} = \frac{3}{4} \cdot \frac{10}{10}$ and $\frac{3}{1} \div \frac{4}{1} = \frac{3 \times 10}{4 \times 10}$ by the transitive property of equality. $\frac{3 \times 10}{4 \times 10} = (3 \times 10) \div (4 \times 10)$ and $\frac{3}{1} \div \frac{4}{1} = (3 \times 10) \div (4 \times 10)$. In general, $a \div b = (a \times 10^y) \div (b \times 10^y)$ is a property when the universe for a and b is R_a and the universe for y is N. We use this property in division computation to change the divisor to a counting number. For example, since $.45 \div 1.5 = (.45 \times 10) \div (1.5 \times 10)$ and $(.45 \times 10) \div (1.5 \times 10) = 4.5 \div 15$, then $.45 \div 1.5 = 4.5 \div 15$. $.45 \div 1.5 = x$ and $4.5 \div 15 = x$ are equivalent conditions. They have the same solution set.

We do not have to think about caret marks or arbitrary rules if we understand what we are doing. When dividing 4.5 by 15, we should think of dividing 45 tenths by 15. Then we can divide as if the dividend were a natural number and express the quotient as a number of tenths. D11 on page 379 shows how to obtain the quotient of 7 and .73 to the nearer hundredth. $7 \div .73 = 700 \div 73$. Since the quotient is to be expressed to the nearer hundredth, 700.00 is used as the dividend. Then we think of 700.00 as 70,000 hundredths and perform the computation as if we were finding the quotient of 70,000 and 73. We write a decimal point in the numeral for the quotient so that it is expressed in hundredths.

In rounding the quotient to the nearer tenth, hundredth, or thousandth, we first think of the number that is the product of $\frac{1}{2}$ and the divisor. If the remainder is greater than that product, we round the quotient up, and if it is less than that product, we round it down.

If the remainder *is* the product of $\frac{1}{2}$ and the divisor, we round to the nearer *even* tenth, hundreth, or thousandth. There is a reason for this. Without this rule, optimistic people might always round up; pessimists might always round down. Suppose that a person has to find the sum of many measures, carrying out his calculations to a given number of significant digits. If he has to round many numbers that are $\frac{1}{2}$ of a hundredth, for example, and he always rounds up, then the sum that he finds is too great. Theoretically, for a great number of examples, if the number is always rounded to the even number choice, then one out of two will be rounded up and one out of two will be rounded down. This results in an answer that is a more accurate measure. This topic is treated in detail in *STM 2*.

From the work in lesson 85, we know that the reciprocal of x is $\frac{1}{x}$. The reciprocal of 2.5 is, then, $\frac{1}{2.5}$. Because $\frac{1}{2.5}$ means $1 \div 2.5$, the quotient of 1 and 2.5 is the reciprocal of 2.5. $1 \div 2.5 = .4$. So .4 is the reciprocal of 2.5.

Instruction notes

■ The students worked with decimal computation in earlier grades. Now that they are more familiar with the properties and characteristics of the decimal system of numeration, they are better equipped to understand the reasons for the rules they learned for computation with decimals.

To find how much time to devote to the first part of this lesson, you could assign "Keeping skillful" exercises 21 through 34 on page 391 as a pretest.

Most classes should be able to complete the exercises through exercise X on page 377 in one

class period. Then "On your own" exercises 1 through 30 may be assigned. On the second day, the lesson can be finished and the rest of the "On your own" exercises assigned.

The work relating to addition and subtraction can be reviewed quickly by discussion and work at the chalkboard. Spend some time developing an understanding of how a decimal point is located in a decimal that expresses a product. Use several examples of multiplication involving mixed numerals and decimals that express the same rational numbers. For example, write on the chalkboard: $\frac{21}{100} \times \frac{13}{10} = \frac{21(13)}{100(10)}$. $\frac{21(13)}{100(10)} = \frac{273}{1000}$. $\frac{21}{100} \times \frac{13}{10} = \frac{273}{1000}$. $.21 \times 1.3 = .273$.

Ask: "Have we expressed a product in two different ways?" [Yes] "Think about the product that is expressed by a common fraction numeral. Is the denominator 100 times 10? Is the product indicated by a fraction whose denominator is 1000?" [Yes] "Now think about the product that is expressed by a decimal. Did we multiply a number of hundredths by a number of tenths?" [Yes] "Is this product expressed in thousandths?" [Yes] "Suppose we were to find the product of two numbers one of which is expressed in hundredths and the other in thousandths. How will the product be expressed?" [In hundred-thousandths]

Before exercise B on page 378, say: "Let's see how this works." Write on the chalkboard: $\frac{2}{3} \div \frac{1}{2} = \frac{2}{3} \times \frac{2}{1}$. $\frac{2}{3} \div \frac{1}{2} = \frac{4}{3}$. Say: "We will multiply both the dividend and the divisor by the same rational number and see if we obtain the same quotient. What is the product of $\frac{2}{3}$ and $\frac{1}{4}$?" [$\frac{2}{12}$] "What is the product of $\frac{1}{2}$ and $\frac{1}{4}$?" [$\frac{1}{8}$] Write: $\frac{2}{12} \div \frac{1}{8} = \frac{2}{12} \times \frac{8}{1}$. $\frac{2}{12} \times \frac{8}{1} = \frac{16}{12}$, or $\frac{4}{3}$. $\frac{2}{12} \div \frac{1}{8} = \frac{4}{3}$. Ask: "Did we find the same quotient when we multiplied both the dividend and the divisor by the same rational number?" [Yes]

Provision for individual differences

■ A class of abler students could cover this lesson in one or one and one half periods. Without using the textbook, review with them addition, subtraction, and multiplication involving decimals. For a review of division, use the book, beginning with page 378.

Use at least two periods for this lesson with an average or ungrouped class. See the "Instruction notes" to find how the lesson might be completed in two days.

For a class of slow learners, devote three periods to this lesson. Teach addition and subtraction in the first period, using many examples on the chalkboard. On the second day, teach multiplication and both division and multiplication by powers of ten. On the third day, teach division, but omit the proof in exercises B through E on page 378. Use several examples to help these students generalize that multiplying both a dividend and a divisor by the same number does not affect the quotient.

7 / 88 Exploring ideas / Pages 382-386

Repeating decimals

Objectives

■ The students learn to convert common fraction numerals to decimals and decimals to common fraction numerals. They learn what terminating and repeating decimals are and how to use decimals to order rational numbers.

Content overview

■ In lesson 85 we show that $\frac{a}{1} \div \frac{b}{1} = \frac{a}{b}$. Because a and b are the natural number mates of $\frac{a}{1}$ and $\frac{b}{1}$, we can think of the rational number $\frac{a}{b}$ as the quotient of a and b, or as $a \div b$. We can use these ideas to convert common fraction numerals to decimals. $\frac{3}{4} = 3.00 \div 4$, because 3.00 is a decimal for the mate of the natural number 3, and 4, in

this case, is a decimal for the mate of the natural number 4.

Any rational number can be expressed as a decimal. If the division process is used to convert a common fraction numeral to a decimal numeral, the decimal that expresses the quotient is either a terminating decimal or a repeating decimal.

A *repeating decimal* is a decimal in which there is an indefinite repetition of the same digit or series of digits as, for example, .333 . . ., .2323 . . ., etc. The pattern of the repeating digits is known as the *repetend* of the decimal. A bar over the digits that repeat in a decimal indicates the repetend. Thus, the decimal for $\frac{1}{3}$ is the symbol $.\overline{3}$, and the decimal for $\frac{5}{6}$ is $.8\overline{3}$. Notice that the repetend may or may not contain all the digits in the decimal.

A *terminating decimal* is a decimal that expresses a fraction whose denominator is a power of 2 or 5 or a multiple of powers of 2 or 5. If, when dividing to convert a common fraction numeral to a decimal, the remainder 0 is obtained at some point in the division process, the quotient can be named by a terminating decimal. Thus, decimals such as .5, .75, 12.25, and 1.003 are terminating decimals. Terminating decimals may be thought of as repeating decimals whose repetend is zero. For example, $.25 = .25\overline{0}$, $1.4 = 1.4\overline{0}$, and so on. Any terminating decimal repeats the digit 0.

There are numbers, such as the square root of 2, that are not rational numbers and that cannot be expressed by a repeating decimal. Such numbers, called irrational numbers, are discussed in unit 11, *STM 2*.

When the division process is used to convert $\frac{5}{7}$, for example, to a decimal numeral, the only possible remainders are 0, 1, 2, 3, 4, 5, and 6. Not all of these remainders necessarily appear in the division process, but when one of the remainders is repeated, we know that the repetend for th decimal has been established. For $\frac{5}{7}$, the remain ders 1, 3, 2, 6, 4, and 5 occur, in this order. Notic that the division process starts with the numera 5 as the first figure in the dividend. Thus, whe the remainder is 5, the cycle of remainders re peats. The corresponding digits in the quotien are 7, 1, 4, 2, 8, and 5, and the decimal for $\frac{5}{7}$ i $.\overline{714285}$. Because the number of possible non-zer remainders is one less than the divisor, the num ber of possible digits in the repetend is alway less than the divisor. In general, for any rationa number of arithmetic $\frac{a}{b}$, there is a repeating deci mal that expresses $\frac{a}{b}$ and whose repetend contain fewer than b digits. Thus, we know that the repe tend for the decimal that expresses $\frac{1}{3}$, for example, will contain either 1 or 2 digits; the repetend for the decimal that expresses $\frac{1}{163}$ will contain a definite number of digits, but the number of digits, in this case, cannot be 163 or greater than 163.

Instruction notes

■ Before exercise N on page 382, ask: "What decimal expresses the number $\frac{1}{2}$? $\frac{2}{4}$? $\frac{3}{6}$? $\frac{50}{100}$?" [.5] "Is $1 \div 2 = .5$? Is $2 \div 4 = .5$? Is $3 \div 6 = .5$? Is $50 \div 100 = .5$?" [Yes] Write these numerals on the chalkboard: $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, $\frac{50}{100}$. Ask: "Does each of these numerals express a fraction that indicates the rational number of arithmetic $\frac{1}{2}$?" [Yes] "How many decimal numerals express the rational number $\frac{1}{2}$?" [One] "To find the decimal that expresses a rational number of arithmetic, must you use the basic fraction of the number?" [No. The same decimal will be obtained, no matter what fraction is used.]

After the paragraph that follows exercise X on page 383, ask: "What decimal expresses $\frac{1}{5}$?" [.2] "$\frac{2}{5}$?" [.4] "$\frac{3}{5}$?" [.6] "If you know that the decimal for $\frac{1}{5}$ is .2, how can you find the decimal for $\frac{4}{5}$?" [Find the product of .2 and 4.] "If you know that the decimal for $\frac{1}{8}$ is .125, how can you find the decimal for $\frac{5}{8}$?" [Find the product of .125 and 5.]

After exercise Z on page 385, write $.33\frac{1}{3}$ on the chalkboard. Say: "Many of us used to write a decimal for $\frac{1}{3}$ in this way. What is the expanded form of this numeral?" $[3(\frac{1}{10^1}) + 3(\frac{1}{10^2}) + \frac{1}{3}(\frac{1}{10^3})]$ Say: "But we really did not mean that the $\frac{1}{3}$ was in thousandths' position. We meant that it was in hundredths' position. Perhaps we did not know then of a better way to write a decimal for $\frac{1}{3}$. You know that the difference of equal numbers is 0. What is the difference of $\frac{1}{3}$ and .3?" $[\frac{1}{30}]$ "Of $\frac{1}{3}$ and .33?" $[\frac{1}{300}]$ "Of $\frac{1}{3}$ and .333?" $[\frac{1}{3000}]$ "You can see the pattern here. Does the difference come closer and closer to 0?" [Yes] "What is the difference of $\frac{1}{3}$ and $.\overline{3}$?" [0]

Before exercise E on page 385, say: "You know that all terminating decimals are also repeating decimals. Are all repeating decimals also terminating decimals?" [No. The decimal $.\overline{3}$, for example, is not a terminating decimal.] "Is the set of all terminating decimals a subset of the set of all repeating decimals?" [Yes]

Provision for individual differences

■ A class of abler students will especially enjoy the ideas presented in this lesson. Have them determine the decimal representation for $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \ldots, \frac{6}{7}$ and compare their results. Similarly, they could compare the decimal representations for $\frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \ldots, \frac{8}{9}$; for $\frac{1}{11}, \frac{2}{11}, \frac{3}{11}, \ldots, \frac{10}{11}$; and so on.

You should expect a class of abler students to cover this lesson in one day.

For an average or ungrouped class, devote two periods to this lesson. Cover the exercises through Z on page 385 on the first day, and assign "On your own" exercises 1 through 18. On the second day, review the work of the previous period, finish the lesson, and assign the remaining "On your own" exercises.

Most slow learners will be unable to master the content of this lesson in less than two class periods. Do not include "On your own" exercises 29 through 40 in their assignment.

7 / Special challenge / Page 387

Objectives

■ The students learn what is meant by triple primes and are challenged to explain why there is only one set of triple primes.

Content overview

■ The idea of triple primes is encountered in the study of a branch of mathematics known as number theory. Triple primes are three primes such that there is exactly one natural number between the first and the second primes and exactly one natural number between the second and third primes. The only set of triple primes is {3, 5, 7}. Because no even number greater than two is prime, all triple primes are odd.

For any other set of three consecutive odd numbers, one of the numbers is a multiple of 3 and, therefore, at least one of the numbers is a composite number. For example, the consecutive odd numbers 7, 9, and 11 are not triple primes.

The numbers 2, 3, and 5 are consecutive prime numbers, but they are not triple primes, because there is no natural number between 2 and 3. The numbers 1, 3, and 5 are consecutive odd numbers, but they are not triple primes, because 1 is not a prime number.

7 / 89 Exploring ideas / Pages 388-391

The density property

Objectives

■ The students study the density property of the set of rational numbers of arithmetic. They learn that there is an infinite set of rational numbers be-

tween any two given rational numbers and that the set of rational numbers does not have a successor property.

Content overview

■ In lesson 59 we presented the betweenness and successor properties of the set of natural numbers. In this lesson we compare the set of rational numbers of arithmetic with the set of natural numbers to find out if the set of rational numbers of arithmetic has either a betweenness or a successor property.

It is always possible to find at least one rational number between any two given rational numbers by using the condition expressed by $(x + y) \div 2$. The universe for x and y is R_a. After x and y are replaced, the expression $(x + y) \div 2$ names a number between the replacements. A rational number between $\frac{3}{100}$ and $\frac{7}{200}$ is $(\frac{3}{100} + \frac{7}{200}) \div 2$, or $\frac{13}{400}$. $\frac{13}{400}$ is between $\frac{3}{100}$ and $\frac{7}{200}$ because $\frac{3}{100} < \frac{13}{400} \wedge \frac{13}{400} < \frac{7}{200}$ is a true statement. The condition may be used again to obtain a number between $\frac{13}{400}$ and $\frac{7}{200}$. $(\frac{13}{400} + \frac{7}{200}) \div 2$, or $\frac{27}{800}$, is between $\frac{13}{400}$ and $\frac{7}{200}$ because $\frac{13}{400} < \frac{27}{800} \wedge \frac{27}{800} < \frac{7}{200}$. The *density property of the set of rational numbers of arithmetic* is: For each replacement of the variables, if $x < y$, there is a number r such that $x < r \wedge r < y$. The universe for each variable is R_a.

Because the condition used in the paragraph above can be applied indefinitely to find numbers between two given rational numbers, there is an infinite set of rational numbers between any two given members of R_a.

Notice that, if the same replacement is made for x and y, then $(x + y) \div 2$ is *not* between x and y. It is not possible to think of a number between $\frac{3}{1}$ and $\frac{3}{1}$.

Observe that, in *STM 1*, the representation of the number line for the set R_a is different from that of the natural-number line. In the picture of the natural-number line, the red dots represent the points associated with natural numbers. In D2 and D3 on page 389, red is used in the pictures of the number line to represent specific points associated with rational numbers of arithmetic and points associated with numbers that are not named in the displays. These representations might incorrectly be interpreted to signify that a rational number of arithmetic is associated with each point in the line. This idea might appear to be justified by the density property of the rational numbers of arithmetic that is emphasized here. Actually, there are points in the line that are not associated with any member of the set of rational numbers. These points are associated with irrational numbers such as π, $\sqrt{2}$, and so on. An introduction to the study of irrational numbers is presented in *STM 2*.

In D2 and D3, observe the arrow at the right of the picture of the line. This arrow emphasizes the fact that R_a does not have a greatest number.

Since no rational number of arithmetic succeeds another, the set of rational numbers of arithmetic does not have a successor property. Thus, $\frac{2}{8}$ is not the successor of $\frac{1}{8}$, because $\frac{3}{16}$ is between $\frac{2}{8}$ and $\frac{1}{8}$; $\frac{3}{16}$ is not the successor of $\frac{1}{8}$, because $\frac{5}{32}$ is between $\frac{1}{8}$ and $\frac{3}{16}$; and so on.

The lack of a successor property has implications for the tabulation of most subsets of R_a. There can be no tabulation of $\{x | x < \frac{1}{5}\}$ because the second member cannot be named. The first member of $\{x | x < \frac{1}{5}\}$ is 0, of course, but, since 0 does not have a successor in R_a, the "second" member cannot be expressed. Also, the greatest member cannot be named.

In lesson 90 we discuss naming solution sets of conditions for inequality by using standard descriptions. While students have been able to tabulate the solution sets of most conditions up to this point in *STM 1*, in much of their future work in mathematics they will have to write standard descriptions of solution sets.

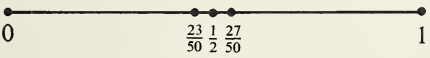
Instruction notes

■ After exercise A on page 388, ask: "In the number line represented in D1, are any points between point 0 and point 1 associated with natural numbers?" [No]

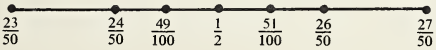
Before exercise F on page 388, ask: "In the set of natural numbers, what is the successor of x?" [$x + 1$] Write on the chalkboard: $a < b \wedge b < a + 1$. U = N. Say: "The solutions of this condition are natural numbers that are between the natural number a and its successor. For any replacement of a, what replacements for b satisfy this condition?" [None. There is no natural number between any natural number and its successor.]

After exercise Q on page 389, ask: "Why do we have to say that the replacement for one variable is less than the replacement for the other variable?" [Because if x and y are replaced by the same number, then $(x + y) \div 2 = x$, and since $x = y$, $(x + y) \div 2$ is not between x and y.]

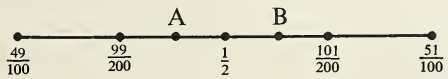
Before exercise V on page 389, write the following on the chalkboard:



Say: "Here is a picture of a segment included in a rational-number line. Now suppose that we magnify the picture of the line between $\frac{23}{50}$ and $\frac{27}{50}$ to obtain the picture below."



Say: "Now suppose that we magnify the picture again for the part between $\frac{49}{100}$ and $\frac{51}{100}$ to obtain another picture."



Ask: "What number is associated with point A?" [$\frac{199}{400}$] "What number is associated with point B?" [$\frac{201}{400}$] "If we kept on magnifying a part of the picture of a rational-number line, would we ever name all the rational numbers between any two rational numbers?" [No]

Provision for individual differences

■ Have your abler students try to prove that if a is between x and y, and b is between a and x, then b is between x and y. The universe for each variable is R_a. Tell them that they may assume that $x < y$. They will also need to use the property that, if $r < s$ and $s < t$, then $r < t$. [If $x < y$ and a is between x and y, then $x < a \wedge a < y$. If b is between a and x, then $x < b \wedge b < a$. If $b < a$ and $a < y$, then $b < y$. Because $x < b \wedge b < y$, b is between x and y.]

With a class of slow learners, you could complete through exercise W on page 389 on the first day and assign "On your own" exercises 1 through 13. On the second day, review the work of the previous day, finish the lesson, and assign "On your own" exercises 14 through 30.

7 / 90 Exploring ideas / Pages 391-396

Solution sets of conditions in one variable

Objectives

■ The students learn to find the solution sets of simple and compound conditions in one variable when the universe is the set of rational numbers of arithmetic. They learn to use a picture of a rational-number line to make a graph of a solution set of a condition.

Content overview

■ All conditions considered in this lesson involve addition or subtraction.

For problems involving natural or counting numbers, objects, such as markers, can be used to represent the numbers and to justify the computation involved in finding the solutions of the conditions. But this procedure cannot be used when rational numbers are involved. Therefore, the relation of subtraction to addition that has been developed in earlier lessons is used to explain the computation needed to find the solution of a condition involving rational numbers.

Since $x - \frac{1}{2} = \frac{1}{4}$ means $x = \frac{1}{2} + \frac{1}{4}$, the solution of each condition is the same number. Because $\frac{2}{3} + x = \frac{5}{6}$ also means $x = \frac{5}{6} - \frac{2}{3}$, the solution of each condition is the same number. Notice that the operation symbol in a sentence does not necessarily indicate what computation should be used to find the solution of the condition.

Two equivalent conditions are used to find the solution of $\frac{3}{5} - x = \frac{1}{10}$. Because $\frac{3}{5} - x = \frac{1}{10}$ means $\frac{3}{5} = x + \frac{1}{10}$ and this in turn means $\frac{3}{5} - \frac{1}{10} = x$, it follows that $\frac{3}{5} - x = \frac{1}{10}$ also means $\frac{3}{5} - \frac{1}{10} = x$. In general, if the replacements for a and b are known, the solution of $a - x = b$ is $a - b$; the solution of $a + x = b$ (or $x + a = b$) is $b - a$; the solution of $x - a = b$ is $a + b$. Each of these solutions can be found by using the addition-subtraction relation.

The solutions of the conditions for equality introduced in this lesson are found by addition and subtraction. The solutions of a condition for inequality are found by first finding the solution of a related condition for equality. The solutions of the condition for inequality are all the members of the universe that are either greater than or less than the solution of the condition for equality. The condition for equality related to $\frac{1}{4} + x < \frac{7}{8}$ is $\frac{1}{4} + x = \frac{7}{8}$. Since the solution of the condition for equality is $\frac{5}{8}$, the solutions of $\frac{1}{4} + x < \frac{7}{8}$ are all members of R_a that are less than $\frac{5}{8}$. Since the set of solutions of the inequality cannot usually be tabulated, it must be expressed by a standard description. One standard description of this solution set is $\{x | \frac{1}{4} + x < \frac{7}{8}\}$. A simpler standard description of the same set is $\{x | x < \frac{5}{8}\}$. The simpler standard description more clearly indicates the membership of the set and should be used.

Graphs that involve pictures of rational-number lines are introduced on page 393. To indicate one member of R_a, the dot that represents the point associated with that number is encircled. Notice that, in graphs B and C in D6 on page 393, bars rather than encircled dots are used to indicate subsets of R_a because rational numbers are dense. It is not always possible to encircle every dot that represents a point that is associated with a solution of a condition for inequality. If a bar has a vertical stroke on the end, the dot through which the stroke passes is contained in the graph of the solution set. If a bar has a curved stroke on the end, the dot immediately outside the stroke is named but is not contained in the graph of the solution set. If the bar extends to the arrowhead, the graph contains all dots indicated in the drawing and all the dots for the points to the right of the last point named.

The graph of the solution set of a compound condition has two bars, one above and one below the picture of the rational-number line. Each bar is the graph of the solution set of one of the simple conditions. The portion of the graph of the universe that has a bar above it as well as a bar below it indicates the graph of the solution set of the compound condition.

Observe that the graphs of four sets are indicated by graph H in D10 on page 395. Graph H includes a graph of R_a, of $\{x | x > \frac{1}{3}\}$, of $\{x | x < \frac{3}{2}\}$, and of $\{x | x > \frac{1}{3} \wedge x < \frac{3}{2}\}$.

The set of non-zero rational numbers of arithmetic was discussed in lesson 85. The letter Z is used as a name of this set in the *STM* program. Z is the universe for the variables in conditions for equivalence.

Instruction notes

■ You may find it necessary to devote two periods to this lesson. On the first day, complete through exercise O on page 393 and assign "On your own" exercises 1 through 13. Finish the lesson on the second day and assign "On your own" exercises 16 through 22 and 26 through 32.

Before exercise A on page 391, you might ask: "How do you find the solution of 3 plus x equals 7?" [Subtract 3 from 7.] "How do you find the solution of x minus 4 equals 15?" [Add 4 to 15.] "How do you find the solution of 10 minus x equals 6?" [Subtract 6 from 10.]

After exercise E on page 391, ask: "How do you know that the conditions expressed by sentences C and D in the display have the same solution?" [Because of the relation of subtraction to addition, $x - \frac{1}{2} = \frac{1}{4}$ means $x = \frac{1}{2} + \frac{1}{4}$.]

After exercise Y on page 392, ask: "Does the symbol for an operation in an open sentence always indicate what computation you use to find the solution of the condition?" [No] "Does it ever indicate what computation to use to find the solution?" [Yes. To find the solution of $x = \frac{1}{8} + \frac{1}{2}$, for example, $\frac{1}{2}$ is added to $\frac{1}{8}$.]

Before exercise H on page 393, write on the chalkboard: $\{x | x + \frac{1}{2} < 2\}$ $\{x | x < 1\frac{1}{2}\}$. Point to the first standard description and ask: "Is this the simplest standard description that can be made for this set?" [No] "Does this description contain a symbol for an operation?" [Yes] "The second standard description on the board describes the same set. Does it contain a symbol for an operation?" [No] "Do you think that the simplest standard description of any set will or will not contain a symbol for an operation?" [It will not.]

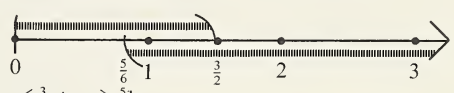
Provision for individual differences

■ A class of abler students could omit the following developmental exercises:

B, C, page 391
F, K through N, page 392
B, C, E, page 393
K, L, page 395
Q, R, S, page 396

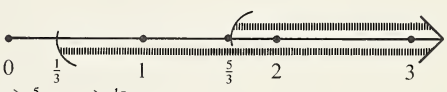
For each graph below, have abler students write a sentence that expresses the compound condition whose solution set is indicated. $U = R_a$. One answer is suggested for each exercise.

1



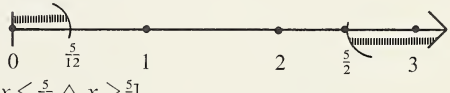
$[x < \frac{3}{2} \wedge x > \frac{5}{6}]$

2



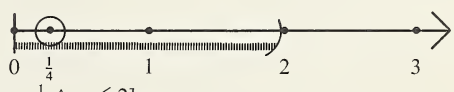
$[x > \frac{5}{3} \wedge x > \frac{1}{3}]$

3



$[x < \frac{5}{12} \wedge x > \frac{5}{2}]$

4



$[x = \frac{1}{4} \wedge x < 2]$

7 / 91 Exploring problems / Pages 397-401

Conditions in one variable for problems

Objectives

■ The students learn to synthesize what they know about problem solving. They use rational numbers as the universe for a variable and find the solution set of a simple or a compound condition in one variable.

Content overview

■ The problems in this lesson are similar to those studied previously in *STM 1*, except that these problems involve rational numbers.

A condition for a problem should reflect the structure of the problem. This means that a condition should indicate the sequence of events described by the problem. This gives us an organized method of solving a problem. If we are told, for example, that Mrs. Jones had $2\frac{1}{4}$ yd. of oilcloth after she bought $\frac{1}{2}$ yd., we can think that she had some oilcloth, acquired another $\frac{1}{2}$ yd., and then had $2\frac{1}{4}$ yd. The condition $x + \frac{1}{2} = 2\frac{1}{4}$ indicates the sequence of events described by the problem. If we know that Jerry had some money, then bought a present that cost \$2.59 and had \$10.91 left, we can use the condition $x - 2.59 = 10.91$. Neither of these sentences that expresses a condition for a problem contains the operation symbol that indicates the computation used to find the solution of the condition. We use the relation that exists between subtraction and addition to find a condition equivalent to the condition for the problem. Since $x + \frac{1}{2} = 2\frac{1}{4}$ also means $x = 2\frac{1}{4} - \frac{1}{2}$, the solution of $x + \frac{1}{2} = 2\frac{1}{4}$ is found by subtracting $\frac{1}{2}$ from $2\frac{1}{4}$.

In general, our problem-solving approach requires careful consideration of the following questions when solving a problem:

1 What is a condition for the problem?

2 What computation is used to find the solutions of the condition?

3 What are the solutions of the condition?

4 What are the answers to the problem?

Notice that for the problem in D4 on page 398, not all the solutions of a condition for the problem are used to obtain answers. While 10.0762 is a solution of the condition, \$10.0762 is not an answer to the problem because no combination of our currency has this monetary value. The idea of selecting a member of the solution set of a condition to obtain an answer to a problem has been previously used in *STM 1*. (For each of the problems in D3 and D4 on page 75 of *STM 1*, the least member of the solution set of the condition for the problem is used to obtain an answer to the problem.)

Observe that the solution set of a condition for a problem such as that in D4 on page 398 can be expressed only by a standard description. The subset of the solution set that is used to obtain answers can be tabulated, but that subset is not the solution set of the condition.

Instruction notes

■ You might begin the lesson by saying: "Read the problem in D1. Suppose this problem read that she bought 3 yd. of oilcloth, then had 12 yd., and you were to find how much she had before she bought the 3 yd. of oilcloth. What is a condition for this problem?" [$x + 3 = 12$.] "How do you find the number that satisfies x plus 3 equals 12?" [Subtract 3 from 12] "What is the answer to the problem?" [9 yd.] "You use the same procedure to solve problems that involve rational numbers that you use to solve problems that involve natural numbers." Then proceed with the exercises on page 397.

For many classes, you might have the students develop a condition for each problem in D3 through D9 without using the developmental exercises. Have the students tell how they can use the relation that exists between subtraction and addition to find the solutions of each condition. When they give the answers to a problem, have them tell if every solution of a condition for that problem can be used to obtain an answer to the problem.

Provision for individual differences

■ A class of abler students could omit all of the developmental exercises except those that deal with the problems in D1, D4, and D9.

A class of slow learners might need two periods to cover this lesson adequately. On the first day, complete through exercise Q on page 398 and assign "On your own" problems 1 through 9, and problems 11, 12, and 13. Finish the lesson on the

econd day and assign problem 10 and problems 4 through 22 of the "On your own" section. For low learners, work out in detail the problems in he displays and give them individual help with heir homework assignment.

/ 92 Exploring ideas / Pages 402-404

Rational-number components of rate pairs

Objectives

The students learn to find the solutions of conditions for equivalence when the components of each rate pair are non-zero rational numbers.

Content overview

Some rate situations involve non-zero rational numbers rather than counting numbers. Now that we have established the ideas of rate pair and rational number, we can use these ideas to solve a wide variety of rate and comparison problems.

This is not a lesson in problem solving. The problem in D1 serves to motivate the lesson and is used only to establish the fact that some rate situations involve rational numbers.

A standard description can be written for a proportional relation that involves rational numbers, but there is little to be gained by introducing this idea to the students at this time. Therefore, neither a standard description nor a tabulation of a proportional relation whose members have non-zero rational number components is considered in this lesson.

For each rate problem, there is a condition for equivalence, such as $\frac{2}{3}/\frac{1}{2} \sim 5/x$. The definition of equivalent ordered pairs can be used to obtain $\frac{2}{3}x = 5(\frac{1}{2})$ from this condition. The ideas developed in lesson 85 are used to find the solution of $\frac{2}{3}x = \frac{5}{2}$. First, $\frac{2}{3}x$ is multiplied by the reciprocal of $\frac{2}{3}$ to obtain the product $\frac{3}{2}(\frac{2}{3}x)$. Since $\frac{2}{3}x = \frac{5}{2}$, the well-defined property of multiplication can be used to obtain $\frac{3}{2}(\frac{2}{3}x) = \frac{3}{2}(\frac{5}{2})$. The associative property of multiplication of rational numbers is next used to obtain $(\frac{3}{2} \cdot \frac{2}{3})x = \frac{3}{2}(\frac{5}{2})$. Since $(\frac{3}{2} \cdot \frac{2}{3})x = x$, $\frac{3}{2}(\frac{2}{3}x) = x$ by the use of the transitive property of equality. Therefore, $x = \frac{3}{2}(\frac{5}{2})$ is obtained, and the solution of this condition is $\frac{15}{4}$. The solution of $x = \frac{3}{2}(\frac{5}{2})$ is the same number as the solution of $\frac{2}{3}/\frac{1}{2} \sim 5/x$.

Rate pairs equivalent to a given rate pair can be found by multiplying or dividing the components of the given rate pair by the same non-zero rational number. In general, $\frac{a}{b} \sim \frac{a \cdot c}{b \cdot c}$, and $\frac{a}{b} \sim \frac{a \div c}{b \div c}$. The universe for a, b, and c is Z. For example, $\frac{1}{4}/2\frac{1}{3} \sim (\frac{1}{4} \cdot \frac{7}{8})/(2\frac{1}{3} \cdot \frac{7}{8})$ and $1\frac{1}{2}/\frac{7}{9} \sim (1\frac{1}{2} \div 2\frac{3}{5})/(\frac{7}{9} \div 2\frac{3}{5})$. This information is useful in finding the solution of a condition such as that expressed in D6. Because $75 = 7.5 \times 10$, the solution of $.2/10 \sim x/75$ is $(7.5 \times .2)$.

Notice that the sets described in "On your own" exercises 21 through 40 are not proportional relations. Each set described contains exactly one member of Z.

Instruction notes

After the introductory paragraph on page 402, say: "Read the problem in D1. Suppose that this mixture contained 7 lb. of peanuts for each 2 lb. of cashews. What rate pair represents this comparison?" [7/2 or 2/7] "Also suppose that we are thinking of a mixture that contains 28 lb. of peanuts and we are to find how many pounds of cashew nuts this mixture contains. What is a condition for the problem I have just described?" [$7/2 \sim 28/x$ or $2/7 \sim x/28$.] "What definition do we use to obtain a condition for equality from $7/2 \sim 28/x$?" [Definition of equivalent ordered pairs] "What condition for equality do we obtain?" [$7x = 28(2)$.] "What is the solution of this condition?" [8] "What is the answer to the prob-

lem?" [8 lb. of cashew nuts] "You are going to learn to use ideas you know to find solutions of conditions for equivalence involving rational numbers."

If necessary, have the students turn to the definition of equivalent ordered pairs given on page 168.

After exercise D on page 402, say: "You know that Z does not contain 0. Think back to the work you did with rate pairs in unit 4. Was the universe for the variable in each condition for equivalence N, or was it C?" [C] "Does C contain 0?" [No]

After exercise H on page 403, write on the chalkboard: $\frac{4}{5}(\frac{5}{4}y) = \frac{4}{5}(\frac{5}{4}y)$. Ask: "Does this sentence mean that a number is equal to itself?" [Yes] "You know that $\frac{5}{4}y$ is equal to $\frac{15}{4}$. Can we replace one $\frac{5}{4}y$ by $\frac{15}{4}$?" [Yes] "What property do we use to make this replacement?" [Well-defined property of multiplication of rational numbers]

Provision for individual differences

■ A class of abler students could omit the following developmental exercises:

N through U, page 403

Have your abler students give a proof that the solution of $\frac{a}{b}(x) = \frac{c}{d}$ is $\frac{cb}{da}$. [$\frac{a}{b}(x) = \frac{c}{d}$. $\frac{b}{a} \cdot \frac{a}{b}(x) = \frac{b}{a} \cdot \frac{c}{d}$. $(\frac{b}{a} \cdot \frac{a}{b})x = \frac{b}{a} \cdot \frac{c}{d}$. $1 \cdot x = \frac{b}{a} \cdot \frac{c}{d}$. $x = \frac{b}{a} \cdot \frac{c}{d}$. $x = \frac{c}{d} \cdot \frac{b}{a}$. $x = \frac{cb}{da}$.]

For the slow learners, emphasize the manipulative aspects of finding solutions of conditions for equality obtained from conditions for equivalence. Do not stress the properties used to find the solutions of the conditions for equality.

Your students might benefit from the use of a chart for the problem in D1. A chart is given below.

Number of pounds of peanuts	Number of pounds of cashew nuts
$1\frac{1}{4}$	$\frac{1}{2}$
$2\frac{1}{2}$	1
5	2
$7\frac{1}{2}$	x

Problems involving rate pairs and rational numbers

Objectives

■ The students learn to find solutions of simple or compound conditions in one or two variables for rate problems that involve rational numbers.

Content overview

■ We think of an *arithmetical mean* as the first component of a rate pair whose second component is 1, and we call it an "average." Also, attention is given to finding sensible answers from the solution set of a condition for a problem.

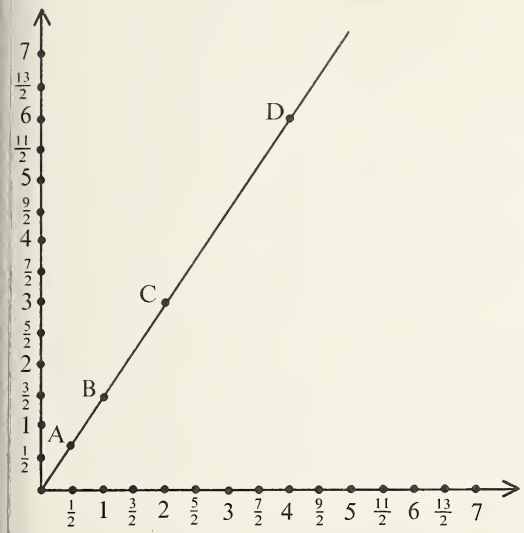
At least two different conditions can be written for a given rate problem. For the problem in D1, both $\frac{1}{2}/\frac{3}{4} \sim n/6$ and $\frac{3}{4}/\frac{1}{2} \sim 6/n$ may be written. However, the condition $\frac{3}{4}/6 \sim \frac{1}{2}/n$ should not be accepted for this problem, even though it has the same solution as the other two. The reason is that the rate pair $\frac{3}{4}/6$ does not represent a comparison that is given in the problem.

Since the first component of $\frac{3}{4}/\frac{1}{2}$ refers to a number of pounds of candy that Kay sold, the first components of other rate pairs used in the condition for the problem must also refer to a number of pounds of candy Kay sold.

The definition of equivalent ordered pairs and also a chart or a graph may be used to find ordered pairs equivalent to a given ordered pair. An example of each is given below.

Number of pounds of candy Kay sold	Number of pounds of candy Agnes sold
$\frac{3}{4}$	$\frac{1}{2}$
$1\frac{1}{2}$	1
3	2
6	4

When one rate pair is given, equivalent rate pairs may be found by multiplying each component of the given rate pair by the same non-zero rational number.



The picture of the diagonal line is a graph of the proportional relation $\{(n, m) | \frac{1}{2}/\frac{3}{4} \sim n/m\}$. The replacements for n are associated with points in the first axis represented in the graph. Point A is associated with the ordered pair $(\frac{1}{2}, \frac{3}{4})$, point B with $(1, \frac{3}{2})$, point C with (2, 3), and point D with (4, 6). Once the graph of $\{(n, m) | \frac{1}{2}/\frac{3}{4} \sim n/m\}$ is made, the amount of candy each girl sold can be quickly found. If Agnes sold 3 lb., then the point in the second axis that is located directly to the left of the point in the line that is directly above point 3 in the first axis is associated with the number of pounds of candy that Kay sold. Of course, when we use a graphical representation, an exact number of pounds cannot always be obtained. For example, if Kay sold $2\frac{7}{16}$ lb., then Agnes sold a little over $1\frac{1}{2}$ lb. In this case, the definition of equivalent ordered pairs should be used to find that Agnes sold $1\frac{5}{8}$ lb.

Some problems in the lesson involve compound conditions. The universe for the variables in these conditions is the Cartesian set $Z \times Z$. Because the replacements for the variable in the condition for equivalence must be non-zero rational numbers, the replacements for the other variable must also be members of Z. To find the solutions of each of these compound conditions, we first find the solution of the simple condition in one variable.

Observe how the solutions of some conditions are used to obtain answers. Mark can buy just 3 pencils for 25¢ when the pencils sell for 8¢ each. If 3 cans of lemonade cost 34¢, one can will cost 12¢, and so on. Notice that an average is a number that has a special interpretation. It is correct, both mathematically and socially, to speak of 34.7 persons or $4\frac{1}{8}$ horses when you are referring to an arithmetic mean.

An arithmetic mean is sometimes simply called "the mean," but this is not precise because "the mean" could refer to a geometric mean, harmonic mean, and so on.

Instruction notes

■ You might begin the lesson by saying: "Read the problem in D1. Suppose that Kay sold 4 lb. of candy for each 3 lb. Agnes sold, and Kay sold 12 lb. You are to find how many pounds Agnes sold. What is a condition for this problem?" [$4/3 \sim 12/x$ or $3/4 \sim x/12$.] "If we use the definition of equivalent ordered pairs, what condition for equality do we obtain?" [$4x = 12(3)$ or $3(12) = x(4)$.] "What is the solution of this condition?" [9] "What is the answer to this problem?" [9 lb. of candy] "You know how to solve rate problems that involve counting numbers. Now you will use similar ideas to solve rate problems that involve rational numbers."

If the students appear to have difficulty with some problems as you develop the lesson, change

the problems so that they involve counting numbers. Then go back and do the problems over again, using the numbers that are given.

You might assign each problem in a display to a row of students, give them a few minutes to work, and then discuss the conditions, solutions, and answers.

You could provide enrichment in your class discussion with a chart or a graph such as those shown in the "Content overview" for this lesson.

Provision for individual differences

■ A class of abler students could omit the following developmental exercises:

A through G, page 404

A through E, page 407

For a class of slow learners, complete through exercise Q on page 405 in the first period and assign "On your own" problems 1, 2, 3, 5, and 6. Finish the lesson in the second period and assign problem 4 and problems 7 through 10. Encourage your slow learners to work all the problems in these short assignments correctly.

---

7 / 94 Exploring ideas / Pages 408-410

Per cents involving rational numbers

Objectives

■ The students learn to find per cents that are equivalent to rate pairs whose components are non-zero rational numbers of arithmetic.

Content overview

■ In unit 4 we discussed per cents as rate pairs that represent comparisons and that have second components of 100. In that unit we discussed only those per cents whose first components are counting numbers.

A per cent is also a rate pair whose second component is the rational number 100 and whose first component is a non-zero rational number of arithmetic.

A per cent that is equivalent to a given rate pair is found by using a condition for equivalence and the definition of equivalent ordered pairs. The per cent equivalent to $1\frac{1}{2}/23$ can be found by using the condition $1\frac{1}{2}/23 \sim x/100$. By the definition of equivalent ordered pairs, $1\frac{1}{2} \times 100 = 23x$. The solution of $150 = 23x$ is found by multiplying 150 by the reciprocal of 23, which yields the same number as dividing 150 by 23. The solution of $150 = 23x$ is $6\frac{12}{23}$. This solution is the first component of $6\frac{12}{23}/100$, or $6\frac{12}{23}\%$.

Observe the distinction between the rate pair a/b and the rate pair $\frac{a}{b}\%$. The rate pair 1/2 is equivalent to 50%, or 50/100, while the rate pair $\frac{1}{2}\%$ is equivalent to $\frac{1}{2}/100$. Obviously, 50/100 is not equivalent to $\frac{1}{2}/100$. The rate pair $\frac{3}{4}\%$ will not be confused with 75% when $\frac{3}{4}\%$ is thought of as $\frac{3}{4}/100$ and 75% is thought of as 75/100.

Instruction notes

■ You might begin the lesson with an example similar to that in D1 and that involves natural numbers only. Put the following on the chalkboard:

	Dollars put into savings account	Income in dollars for one year
Fosters	135	4500
Smiths	477	9540
Baxters	65	6500
Olsons	140	7000

Have students use rate pairs to represent comparisons of the savings and income of each family. Then have them use conditions for equivalence to find the per cents that represent these same comparisons. Let them use the per cents to determine which families saved proportionally more or less of their income. Then say: "Suppose that a family saved $2\frac{1}{2}\%$ of its income. $2\frac{1}{2}\%$ means $2\frac{1}{2}/100$. $2\frac{1}{2}\%$ is a rate pair. What is the second component of the rate pair $2\frac{1}{2}\%$?" [100] "What

is the first component?" [$2\frac{1}{2}$] "If a family saves $2\frac{1}{2}\%$ of its income, it saves how many dollars for each 100 dollars earned?" [$2\frac{1}{2}$] "Now you see that a rational number like $2\frac{1}{2}$ can be a first component of a per cent."

In exercises 17 through 28 in the "On your own" exercises, it is sometimes necessary to express the first component of a per cent as a decimal to the nearer hundredth. You may find that the development below will be helpful to you in a discussion before you assign these exercises. For example, to find a rate pair equivalent to 5.174/7.35 whose first component is expressed as a decimal to the nearer hundredth, the condition $5.174/7.35 \sim x/100$ may be used.

(a) Using the definition of equivalent ordered pairs, we obtain $517.4 = 7.35x$.

(b) Multiplying by the reciprocal of 7.35, we obtain $\frac{1}{7.35} \cdot (517.4) = x$.

(c) The quotient property tells us that $x \div y = x \cdot \frac{1}{y}$ and that $x \cdot \frac{1}{y} = x \div y$. Thus, $\frac{1}{7.35} \cdot (517.4) = 517.4 \div 7.35$. The quotient of 517.4 and 7.35 to the nearer hundredth is 70.39. Thus, $5.174/7.35 \sim 70.39/100$ and $70.39/100 = 70.39\%$.

Provision for individual differences

■ All classes should be able to cover this lesson in one period. A class of abler students could complete it as a self-study assignment. For slow learners, show repeatedly how the solution of a condition for equivalence is found.

---

7 / 95 Exploring problems / Pages 411-414

Problems involving per cents and rational numbers

Objectives

■ The students learn to solve rate problems that involve per cents and rational numbers. They learn to find a per cent of increase and a per cent of decrease.

Content overview

■ A wide range of social situations involve per cents. Besides per cent of increase and per cent of decrease, there are rates of commissions, tax rates, and rates of interest, to name a few. This lesson, as well as certain other lessons on problem solving, presents some of the practical applications of mathematical ideas.

A sentence that expresses a condition for equivalence for a per cent problem is of the general form $a/b \sim x/100$. In the problem in D1 on page 411, the replacements for a and b are named. The problem in D2 gives the replacements for b and x. The problem in D3 on page 412 names the replacements for a and x. For a per cent problem that involves a compound condition, one of the simple conditions is always expressed by a sentence of the form $a/b \sim x/100$.

We can think of a per cent of increase or decrease as a rate pair that is equivalent to another rate pair whose first component is the number that is the increase or decrease, and whose second component is the original number that is either increased or decreased. In the problem described in D6 on page 412, the number that is the increase is not named, but it is the solution of the condition $500 - 240 = x$. The rate pair $x/240$ represents the comparison of the increase to the number that is increased. The rate pair $y/100$ is the per cent of increase. Notice that y is a variable for the number that is the increase for each 100 persons enrolled last year. Thus, the compound condition for this problem is $y/100 \sim x/240 \wedge 500 - 240 = x$.

Instruction notes

■ Most students will be able to find a condition for each of the problems in D1, D2, and D3 on pages 411 and 412 without using the related de-

velopmental exercises. Have them tabulate the solution set of each condition and give the answer to each problem. Then discuss the developmental exercises beginning with A on page 412.

Before exercise H on page 412, say: "A per cent of increase is represented by a rate pair that is equivalent to a special rate pair. The first component of the special rate pair is the number that is the increase. The second component of the special rate pair is the number that is increased."

After the class has completed the developmental exercises, review the idea that a per cent of increase or decrease is a per cent that is equivalent to a special rate pair. The special rate pair represents a comparison of the number that is the increase or decrease with the number that is increased or decreased.

Provision for individual differences

■ Abler students could complete this lesson as a self-study assignment. Slow learners will be helped by class discussion of "On your own" problems 1, 2, 7, 13, and 15 through 18.

---

7 / 96 Exploring problems / Pages 416-418

Problems involving comparisons with one

Objectives

■ The students learn to solve problems that involve conditions for equivalence in which the second component of one rate pair is 1.

Content overview

■ In the previous lesson we compared one number with another by means of per cents. The problems in this lesson also involve the comparison of one number with another. Now, however, we represent these comparisons by means of rate pairs whose second component is 1 rather than 100.

Problems involving comparisons with 1 were encountered in lesson 44. Phrases such as "4 times as many" were used to indicate comparisons represented by rate pairs whose second component is 1. In this lesson, phrases such as "$\frac{1}{3}$ of" and ".5 as many" are also used to indicate that two numbers are being compared and that a rate pair whose second component is 1 represents this comparison. Thus, for problem A in D1, the 11-year life expectancy of a dog is $\frac{1}{3}$ of the 33-year life expectancy of a horse. In other words, 11/33 is the same comparison as $\frac{1}{3}/1$. $\frac{1}{3}/1$ means that for every 1 year that a horse is expected to live, a dog is expected to live $\frac{1}{3}$ of a year.

Of course, the life expectancy of a dog cannot literally be a fraction of the life expectancy of a horse. In this case, "fraction of" is used idiomatically; the phrase "fraction of," as used in the problem in D1, is used in a social sense rather than in a mathematical sense. What is meant is that the student is to find what part of a year a dog is expected to live for each year of a horse's life expectancy. What is meant at a mathematical level is that the student is to find a rational number that is the first component of a rate pair whose second component is 1 such that the rate pair is equivalent to another rate pair whose first component is the number of years of a dog's life expectancy and whose second component is the number of years of a horse's life expectancy.

Observe that the true statement $11/33 \sim \frac{1}{3}/1$ is not the same as the true statement $11/33 \sim 1/3$. For the problem in D1, the rate pair $\frac{1}{3}/1$ represents the idea that a dog is expected to live $\frac{1}{3}$ of a year for each year a horse is expected to live. The rate pair 1/3 represents the idea that a dog is expected to live 1 year for each 3 years a horse is expected to live.

The answers to abstract problems of this type are found in a similar manner. The problem "3.6 is .4 of what number?" really asks "What is the

second component of a rate pair whose first component is 3.6 when that rate pair is equivalent to .4/1?"

The problems in this lesson are examples of the traditional "three cases of fractions." The student uses the mathematics he knows to solve these problems, and thus it is not necessary for him to memorize rules that are difficult to understand. All the problems in this lesson may be solved by using a condition of the form $a/b \sim c/1$. At least two of the three variables, a, b, and c, are replaced by numbers given in a problem.

The idiomatic language used in this lesson is consistent with the usage in this country. Attention is given in the lesson to the meaning of the idiom.

Instruction notes

■ You could begin this lesson by saying: "Turn back to page 182 and read the problem in D5. Do you have to find how many votes Janet received for each vote Sue received?" [Yes] "What is a condition for this problem?" [$39/13 \sim x/1$.] "Suppose that we change this problem. We will include the information given, but we will ask a different question. Suppose the question is: 'The votes Sue received are what fraction of the votes Janet received?' Does this question mean that we are to find what part of a vote Sue received for each vote Janet received?" [Yes] "What rate pair represents this comparison?" [$x/1$] "Does the second component of this rate pair refer to a number of votes Janet received?" [Yes] "Does the second component of 13/39 refer to a number of votes Janet received?" [Yes] "What is a condition for this problem?" [$x/1 \sim 13/39$.] "What is the solution of the condition? The universe for x is Z." [$\frac{1}{3}$] "What is the answer to the problem?" [Sue received $\frac{1}{3}$ as many votes as Janet.] "In this lesson you will learn to solve both abstract problems and problems having social applications that involve comparisons with 1."

Before exercise F on page 417, say: "You can also use a chart to help you find rate pairs equivalent to 11/33." Make this chart on the chalkboard:

11	33

Ask: "Name some rate pairs that are equivalent to 11/33." [22/66, 33/99, $\frac{11}{2}/\frac{33}{2}$, etc.] "If you multiply or divide both components of a rate pair by a non-zero rational number, do you obtain an equivalent rate pair?" [Yes] Extend the chart to look like this:

11	33
22	66
33	99
$\frac{11}{2}$	$\frac{33}{2}$

Ask: "A dog's life expectancy is how many years for each $\frac{33}{2}$ years of a horse's life expectancy?" [$\frac{11}{2}$] "How can we obtain an ordered pair whose second component is 1 from 11/33?" [Divide each component of 11/33 by 33.] Write $\frac{1}{3}/1$ on the chart. Ask: "How can we obtain an ordered pair whose second component is 100 from 11/33?" [Multiply each component of 11/33 by $3\frac{1}{33}$ or by $\frac{100}{33}$.] "What comparison does the rate pair $33\frac{1}{3}/100$ represent for problem A in D1?" [The number of years of a dog's life expectancy for each 100 years of a horse's life expectancy.] "Is a dog's life expectancy $33\frac{1}{3}\%$ of a horse's life expectancy?" [Yes]

As the class discusses exercise M on page 417, be sure the students understand that we are not literally talking about $\frac{3}{4}$ of a satellite. In this problem, "$\frac{3}{4}$ of a satellite" means a rate of $\frac{3}{4}$ of a Saturn satellite for each Jupiter satellite.

Provision for individual differences

■ A class of abler students could do the work in this lesson on a self-study basis. After the students have finished the lesson, review, for each problem in the displays, the condition, the method of finding the solution, and the answer.

The abler students might also consider the following exercises:

1 $\frac{1}{3}$ is $\frac{1}{3}$ of what number? [1] $\frac{1}{n}$ is $\frac{1}{n}$ of what number? [1]

2 $\frac{2}{3}/1$ is equivalent to x/y. What per cent is equivalent to x/y? [$66\frac{2}{3}\%$] If y is replaced by 60, what must x be replaced by to obtain a true statement? [40] If x is replaced by 60, what must y be replaced by to obtain a true statement? [90]

3 5 is $\frac{1}{5}$ of what number? [25] n is $\frac{1}{n}$ of what number? [n^2]

4 Sam has read $\frac{3}{8}$ of the pages of a book. What per cent of the book has he read? [$\frac{3}{8}/1 \sim x/100$. $\{37\frac{1}{2}\}$ $37\frac{1}{2}\%$]

5 Betty has read $\frac{3}{5}$ of a 465-page book. How many pages must she read to complete the book? [$\frac{3}{5}/1 \sim x/465 \wedge 465 - x = y$. $\{(279, 186)\}$ 186 pages]

For a class of slow learners, carefully discuss the developmental exercises. You might also work the first three or four "On your own" problems at the chalkboard. Do not assign "On your own" problems 16 through 21 to these students.

---

7 / 97 Exploring problems / Pages 419-422

Per cents related to rate pairs with second components of one

Objectives

■ The students learn to use rate pairs such as $\frac{1}{3}/1$ for $33\frac{1}{3}\%$, $\frac{3}{8}/1$ for $37\frac{1}{2}\%$, and $\frac{2}{3}/1$ for $66\frac{2}{3}\%$ in conditions for equivalence for rate problems.

Content overview

■ For computational ease, it is often desirable to think of rate pairs whose second components are 1 as equivalent to certain per cents. Because $\frac{1}{6}/1 \sim 16\frac{2}{3}/100$, the rate pair $\frac{1}{6}/1$ may replace $16\frac{2}{3}/100$ in a condition for equivalence. The rate pairs $\frac{1}{6}/1$ and $16\frac{2}{3}\%$ are both members of the same proportional relation. It is helpful to know the following true statements:

$\frac{1}{8}/1 \sim 12\frac{1}{2}/100$	$\frac{3}{5}/1 \sim 60/100$
$\frac{1}{6}/1 \sim 16\frac{2}{3}/100$	$\frac{5}{8}/1 \sim 62\frac{1}{2}/100$
$\frac{1}{4}/1 \sim 25/100$	$\frac{2}{3}/1 \sim 66\frac{2}{3}/100$
$\frac{1}{3}/1 \sim 33\frac{1}{3}/100$	$\frac{3}{4}/1 \sim 75/100$
$\frac{3}{8}/1 \sim 37\frac{1}{2}/100$	$\frac{4}{5}/1 \sim 80/100$
$\frac{2}{5}/1 \sim 40/100$	$\frac{5}{6}/1 \sim 83\frac{1}{3}/100$
$\frac{1}{2}/1 \sim 50/100$	$\frac{7}{8}/1 \sim 87\frac{1}{2}/100$

Although authors in recent years have treated rate pairs and rational numbers as being the same, we make a distinction between them. A per cent should not be regarded as being equal to a rational number; that is, 50% is not equal to the rational number $\frac{1}{2}$, because 50% is the ordered pair (50, 100), and the rational number $\frac{1}{2}$ is an infinite set of ordered pairs of equivalent fractions. 50% does not name a number, but $\frac{1}{2}$ does. However, 50% is equivalent to the rate pair $\frac{1}{2}/1$ because $50 \cdot 1 = \frac{1}{2} \cdot 100$.

Suppose the number that is $87\frac{1}{2}\%$ of 96 is to be found. The condition $87\frac{1}{2}/100 \sim x/96$ can be used. Since $\frac{7}{8}/1 \sim 87\frac{1}{2}/100$ is a true statement, $87\frac{1}{2}/100$ can be replaced in the condition by $\frac{7}{8}/1$ to obtain $\frac{7}{8}/1 \sim x/96$. The definition of equivalent ordered pairs is used to obtain $\frac{7}{8} \cdot 96 = x \cdot 1$. The condition $\frac{7}{8}/1 \sim x/96$ can be used for $87\frac{1}{2}/100 \sim x/96$ because the conditions are equivalent. The conditions are not the same, but equivalent conditions have the same solutions.

Notice that the phrase "the number that is $87\frac{1}{2}\%$ of 96" is an idiom. Mathematically, there is no such thing as a rate pair of a number (or a fraction of a number, either). The question "What

number is $87\frac{1}{2}\%$ of 96?" really asks: "What number is the first component of a rate pair whose second component is 96 when that rate pair is equivalent to $87\frac{1}{2}/100$?"

One way to find a rate pair that is equivalent to $62\frac{1}{2}/100$ and that has a second component of 1 is as follows: $62\frac{1}{2}/100 = 62.5/100$. Each component of 62.5/100 can be divided by 100 to obtain the equivalent rate pair .625/1. Since $.625/1 = \frac{5}{8}/1$, then $62\frac{1}{2}/100 \sim \frac{5}{8}/1$. The definition of equivalent ordered pairs can be used to show that $62\frac{1}{2}/100$ and $\frac{5}{8}/1$ are equivalent.

Content overview

■ Begin the lesson by writing on the chalkboard: 10%, 25%, 40%, 50%, 80%. Say: "You will find it helpful to know some rate pairs with second components of 1 that are equivalent to certain per cents. Let us make a chart for each of the rate pairs expressed on the board." Write the following on the chalkboard:

10	100

25	100

40	100

50	100

80	100

Ask: "By what number can 100 be divided to obtain a quotient of 1?" [100] "If we also divide 10 by 100, do we obtain a rate pair equivalent to 10%?" [Yes] "What rate pair do we obtain?" [$\frac{10}{100}/1$] "What is the basic fraction of $\frac{10}{100}$?" [$\frac{1}{10}$] "Is $\frac{1}{10}/1$ equivalent to 10/100?" [Yes] Use a similar procedure to develop the idea that 25% is equivalent to $\frac{1}{4}/1$, 40% to $\frac{2}{5}/1$, 50% to $\frac{1}{2}/1$, and 80% to $\frac{4}{5}/1$. Then begin the lesson in the book.

After exercise V on page 420, say: "Suppose we want to find a rate pair with a second component of 1 that is equivalent to $83\frac{1}{3}\%$." Write on the board: $83\frac{1}{3}/100$. Say: "You know that you can divide each component by 100 to obtain an equivalent rate pair. Let us make a chart to see if there is another method. Will we obtain an equivalent rate pair if we multiply each component by 3?" [Yes] "What rate pair do we obtain?" [250/300] Write on the board:

$83\frac{1}{3}$	100
250	300

Now say: "We know that $83\frac{1}{3}/100$ is equivalent to 250/300. How can we obtain a rate pair with a second component of 1 that is equivalent to 250/300?" [Divide each component by 300.] "What rate pair do we obtain?" [$\frac{250}{300}/1$] "What is the basic fraction of $\frac{250}{300}$?" [$\frac{5}{6}$] "Is $\frac{5}{6}/1$ equivalent to 250/300?" [Yes] "Is it equivalent to $83\frac{1}{3}\%$?" [Yes] You might have your students follow a similar procedure to find that $8\frac{1}{3}\%$ is equivalent to $\frac{1}{12}/1$.

Provision for individual differences

■ A class of abler students could omit all the developmental exercises on page 419.

Abler students might consider the following exercises:

1 Mary's age, 15, is $41\frac{2}{3}\%$ of her father's age. How old is her father? [$15/x \sim 41\frac{2}{3}/100$, or $15/x \sim \frac{5}{12}/1$. {36} He is 36 years old.]

2 Frank's height is 64 in. His sister's height is $93\frac{3}{4}\%$ of Frank's height. What is Frank's sister's height in inches? [$x/64 \sim 93\frac{3}{4}/100$, or $x/64 \sim \frac{15}{16}/1$. {60} 60 in.]

3 What number is $\frac{1}{2}$ of 50% of 80? Use a compound condition. [$\frac{1}{2}/1 \sim x/y \wedge 50/100 \sim y/80$ or $\frac{1}{2}/1 \sim x/y \wedge \frac{1}{2}/1 \sim y/80$. {(20, 40)} 20]

Encourage the slow learners, and give them extra practice in finding rate pairs with second components of 1 that are equivalent to per cents such as $16\frac{2}{3}\%$, 40%, $33\frac{1}{3}\%$, and so on. You might suggest that these students memorize some of the more common equivalents.

Supplementary material

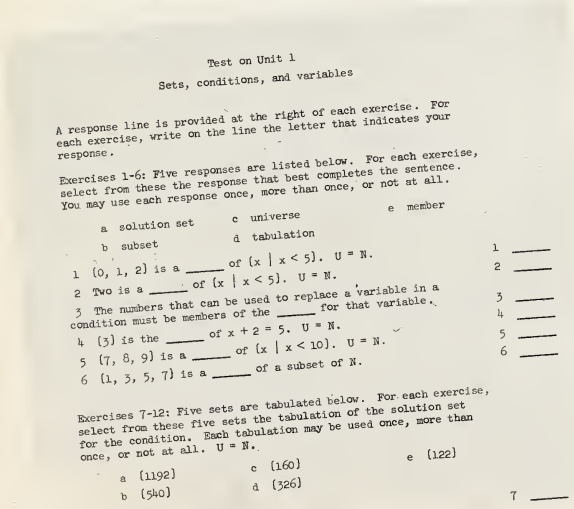
Supplementary tests (end-of-unit and end-of-book)
Properties taught in *STM 1*
Bibliography for teachers
Bibliography for students
Responses to Special Challenges
Index

Supplementary tests

■ The problem of evaluating student achievement confronts every classroom teacher. We have provided a variety of tests that should meet your needs. A number of different tests appear in the student's book. These are described on page 31 of this *Teaching Guide*. Because you may want to use tests that are not in the student's book, we include here suggested test items for each of the seven units and an end-of-book test.

You may use each test as it appears here, or as a framework for a test of your own. In either case, be sure that sufficient time is allowed, so that all students can complete the tests. You can reproduce each test in quantity for class use, or you can write the items on the chalkboard.

A portion of a suggested test form is shown below. Notice that this test form utilizes a response line at the right of each exercise that makes it easy to score the test with a scoring key. You may choose to eliminate the response line in the test booklets you prepare and provide students with separate answer sheets. You can then save the test booklets for future use.



Test on Unit 1

Sets, conditions, and variables

A response line is provided at the right of each exercise. For each exercise, write on the line the letter that indicates your response.

Exercises 1-6: Five responses are listed below. For each exercise, select from these the response that best completes the sentence. You may use each response once, more than once, or not at all.

a solution set c universe e member
b subset d tabulation

1 {0, 1, 2} is a _____ of {x | x < 5}. U = N. 1 ____
2 Two is a _____ of {x | x < 5}. U = N. 2 ____
3 The numbers that can be used to replace a variable in a condition must be members of the _____ for that variable. 3 ____
4 {3} is the _____ of x + 2 = 5. U = N. 4 ____
5 {7, 8, 9} is a _____ of {x | x < 10}. U = N. 5 ____
6 {1, 3, 5, 7} is a _____ of a subset of N. 6 ____

Exercises 7-12: Five sets are tabulated below. For each exercise, select from these five sets the tabulation of the solution set for the condition. Each tabulation may be used once, more than once, or not at all. U = N.

a {1192} c {160} e {122}
b {540} d {326}

7 535 + x = 657. 7 ____

Test on unit 1

Sets, conditions, and variables

Exercises 1-6 Five responses are listed below. For each exercise, select from these the response that best completes the sentence. You may use each response once, more than once, or not at all.

a solution set c universe e member
b subset d tabulation

1 {0, 1, 2} is a ~~~~ of $\{x \mid x < 5\}$. U = N.
2 Two is a ~~~~ of $\{x \mid x < 5\}$. U = N.
3 The numbers that can be used to replace a variable in a condition must be members of the ~~~~ for that variable.
4 {3} is the ~~~~ of $x + 2 = 5$. U = N.
5 {7, 8, 9} is a ~~~~ of $\{x \mid x < 10\}$. U = N.
6 {1, 3, 5, 7} is a ~~~~ of a subset of N.

Answers: 1-b, 2-e, 3-c, 4-a, 5-b, 6-d

Exercises 7-12 Five sets are tabulated below. For each exercise, select from these five sets the tabulation of the solution set for the condition. Each tabulation may be used once, more than once, or not at all. U = N.

a {1192} c {160} e {122}
b {540} d {326}

7 $535 + x = 657$. 10 $a - 433 = 759$.
8 $y - 535 = 657$. 11 $t + 107 = 433$.
9 $243 - n = 83$. 12 $350 + 190 = d$.

Answers: 7-e, 8-a, 9-c, 10-a, 11-d, 12-b

Exercises 13-22 If a sentence expresses a true statement, write "T." Write "F" if the sentence expresses a false statement.

13 Any three points are coplanar points.
14 Point A is a member of $\{X \mid \overline{AX} \cong \overline{AB}\}$.

15 Line AB is a subset of plane ABC.
16 $\overline{AB}$ is a radius of $\{X \mid \overline{AX} \cong \overline{AB}\}$.
17 $\overleftrightarrow{AB}$ separates plane ABC into two half-planes.
18 Point B is a member of $\{X \mid \overline{AX} \cong \overline{AB}\}$.
19 If point D is in $\overleftrightarrow{AB}$, then point D must be between A and B.
20 Plane ABC is the only plane that includes $\overleftrightarrow{AB}$.
21 Point A separates $\overleftrightarrow{AB}$ into two half-lines.
22 If points G and H are in circle A, then chord GH is a subset of circle A.

Answers: 13-T, 14-F, 15-T, 16-T, 17-T, 18-T, 19-F, 20-F, 21-T, 22-F

Exercises 23-26 Five open sentences are listed below. For each exercise, select from the list the sentence that expresses a condition for the problem. Each open sentence may be used once, more than once, or not at all.

a $x - 23 = 10$. **c** $x + 10 = 23$. **e** $23 + 10 = x$.
b $x - 10 = 23$. **d** $23 - x = 10$.

23 Fred caught 23 fish. He gave some of these fish to a neighbor and then had 10 fish left. How many fish did Fred give to his neighbor?
24 Bill had 23 model airplanes. After he bought 10 more model airplanes, he had as many model planes as Jim. How many model airplanes does Jim have?
25 Ten years from now Jack will be 23 years old. How old is Jack now?
26 From what number can you subtract 23 so that the difference is 10?

Answers: 23-d, 24-e, 25-c, 26-a

Exercises 27-30 Five open sentences are listed below. For each exercise, select from the list the sentence that expresses a condition for the problem. Each open sentence may be used once, more than once, or not at all.

a $x - 23 > 10$. **c** $23 + 10 > x$. **e** $x - 10 < 23$.
b $x + 10 < 23$. **d** $x - 10 > 23$.

27 John had some sheets of colored paper. He used 10 of these sheets to make some posters and then had fewer than 23 sheets left. How many sheets of colored paper did John have to begin with?
28 Mary has fewer than 23 nickels and dimes in all. She has 10 nickels. How many dimes can Mary have?
29 23 can be subtracted from what numbers so that each difference is greater than 10?
30 To what numbers can you add 10 so that each sum is less than 23?

Answers: 27-e, 28-b, 29-a, 30-b

Exercises 31-43 Four responses are listed for each exercise. Select the response that best answers the question in the exercise or that correctly completes the exercise.

31 $\overline{AB} < \overline{BC}$ and $\overline{BC} < \overline{AC}$. Which sentence below expresses a true statement?

a $\overline{AB} > \overline{AC}$. **c** $\overline{AB} < \overline{AC}$.
b $\overline{AB} \cong \overline{AC}$. **d** $\overline{BC} > \overline{AC}$.

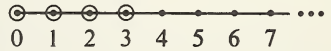
32 The solution set of which condition expressed below is an infinite set? U = N.

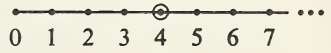
a $294 - n > 5$. **c** $n - 7 = 83$.
b $45 + n < 76$. **d** $n + 3 > 7$.

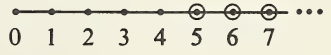
33 Twenty-nine is a solution of which condition expressed below? U = N.

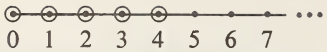
a $21 \div 8 > y$. **c** $65 - y < 36$.
b $y \div 60 < 90$. **d** $y - 6 > 23$.

34 Which of the following four graphs is a graph of the solution set of $x + 4 < 8$? U = N.

a


b


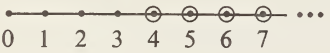
c


d 

35 A standard description of the solution set of $41 + x < 59$ is ~~~. U = N.

a $\{x|x < 18\}$ c $\{x|x > 18\}$
b $\{x|x < 100\}$ d $\{x|x > 100\}$

36 This graph is a graph of which of the sets named below? U = N.



a $\{y|y + 1 > 3\}$ c $\{y|y > 4\}$
b $\{y|y + 1 > 4\}$ d $\{y|y < 3\}$

37 Which of the following names a subset of $\{x|x + 5 < 14\}$? U = N.

a $\{x|x + 6 > 10\}$ c $\{x|12 - x > 5\}$
b $\{x|2x = 18\}$ d $\{x|x + 3 = 3 + x\}$

38 A tabulation of $\{x|63 + x > 79\}$ is ~~~. U = N.

a {17, 18, 19, 20} c {16}
b {17, 18, 19, . . .} d {0, 1, 2, . . ., 15}

39 Points K and L are not the same point. Together they are contained in ~~~ lines.

a exactly one c exactly three
b exactly two d infinitely many

40 A geometric figure that is the boundary of two half-spaces is a ~~~.

a line b circle c plane d half-plane

41 What name do we give to the set that has no members?

a finite set c empty set
b infinite set d subset

42 Which one of the sentences below can you use to help you find the solution set of $28 - x > 19$?

a $28 + x = 19$. c $x - 19 = 28$.
b $28 - x = 19$. d $x = 28 + 19$.

43 A geometric figure that is determined by three noncollinear points is a ~~~.

a plane b segment c line d chord

Answers: 31-c, 32-d, 33-b, 34-a, 35-a, 36-b, 37-c, 38-b, 39-a, 40-c, 41-c, 42-b, 43-a

Intersection and union of sets

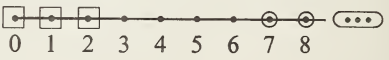
Exercises 1-6 Five sets are tabulated below. For each exercise, select from these five sets the tabulation of the set named. Each tabulation may be used once, more than once, or not at all.

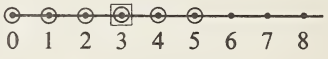
a {1, 2, 3, 4, 5} c {2, 3} e { }
b {2, 3, 4} d {3}

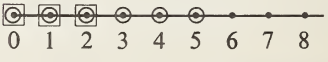
1 The intersection of {1, 2, 3} and {2, 3, 4, 5}
2 The union of {1, 2, 3} and {2, 3, 4, 5}
3 $\{3, 4, 5, \ldots\} \cap \{0, 1, 2, 3\}$
4 $\{2, 3, 4\} \cup \{\ \}$
5 $\{2, 3, 4\} \cap \{\ \}$
6 $\{2, 3, 4\} \cup \{2, 3, 4\}$

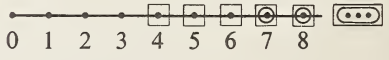
Answers: 1-c, 2-a, 3-d, 4-b, 5-e, 6-b

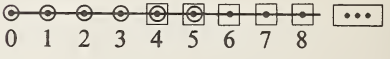
Exercises 7-11 Five graphs are shown below. For each exercise, select from the five graphs the graph of the set expressed. Each graph may be used once, more than once, or not at all. U = N.

a 

b 

c 

d 

e 

7 $\{n|n < 6 \wedge n > 3\}$
8 $\{n|n < 6 \wedge n = 3\}$
9 $\{n|n > 6 \wedge n > 3\}$

10 $\{n|n+2<8\} \cap \{n|n-2>1\}$
11 $\{n|n<6 \wedge n<3\}$

Answers: 7-e, 8-b, 9-d, 10-e, 11-c

Exercises 12-24 If a sentence expresses a true statement, write "T." Write "F" if the sentence expresses a false statement.

12 The intersection of set F and set G is a subset of the union of sets F and G.
13 Any two disjoint planes are parallel.
14 If two lines intersect, they determine exactly two planes.
15 A circle is a simple polygon.
16 The intersection of sets A and B is a subset of set A.
17 $\angle ABC$ is a subset of triangle ABC.
18 $\angle ABC$ is the same angle as $\angle ACB$.
19 A triangle is a simple closed curve.
20 The intersection of the sides of $\angle DEF$ is {E}.
21 $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$.
22 A right angle is a subset of the union of two perpendicular lines.
23 $\overrightarrow{AB} \cap \overleftrightarrow{AB} = \overline{AB}$.
24 The union of two angles that are a linear pair of angles is a ray.

Answers: 12-T, 13-T, 14-F, 15-F, 16-T, 17-F, 18-F, 19-T, 20-T, 21-T, 22-T, 23-F, 24-F

Exercises 25-30 Five responses are listed below. For each exercise, select the response that most clearly expresses the idea described. Each response may be used once, more than once, or not at all.

a ray
b line
c empty set
d triangle
e a set whose only member is one point

25 The intersection of two planes that meet
26 The intersection of a plane and a line that meet when the line is not included in the plane
27 The union of a half-line and its boundary
28 The union of the three segments determined by three noncollinear points
29 The intersection of two skew lines
30 The intersection of a circle and a radius of the circle

Answers: 25-b, 26-e, 27-a, 28-d, 29-c, 30-e

Exercises 31-41 Four responses are listed for each exercise. Select the response that best answers the question in the exercise or that correctly completes the exercise.

31 Suppose that the union of sets R and S is the empty set. What do you know about sets R and S?

a Sets R and S meet.
b R contains at least one member that is not in S.
c Each set is the empty set.
d S contains at least one member that is not in R.

32 The intersection of line MN and plane MNP is ~~~.

a { }
b point M
c plane MNP
d line MN

33 What is the greatest number of right angles a triangle may have?

a 0 b 1 c 2 d 3

34 ~~~ is a tabulation of the solution set of $x-29<13 \wedge 5+x>35$. U = N.

a {17, 18, 19, . . ., 24}
b {17, 18, 19, . . ., 41}
c {31, 32, 33, . . ., 41}
d {31, 32, 33, . . .}

35 Points A, B, and C are noncollinear points. Point D is between point A and point C. Which of the following is the name of a linear pair of angles?

a $\angle ABD$ and $\angle CDB$
b $\angle ADB$ and $\angle BDC$
c $\angle BDA$ and $\angle BCD$
d $\angle BDC$ and $\angle ABD$

36 In triangle ABC, $AB \cong BC$, and $BC > AC$. What kind of triangle is triangle ABC?

a A scalene triangle
b An isosceles triangle
c An equilateral triangle
d A right triangle that is also a scalene triangle

Answers: 31-c, 32-d, 33-b, 34-c, 35-b, 36-b

Use the following problem for exercises 37, 38, and 39:

In 15 years, Mr. Smith will be more than 52 years old. Six years ago he was less than 46 years old. How old can Mr. Smith be now?

37 Which of the following expresses a condition for the problem? U = C.

a $n + 15 < 52 \wedge n - 6 < 46$.
b $n + 15 > 52 \wedge n - 6 > 46$.
c $n + 15 > 52 \wedge n - 6 < 46$.
d $52 - n > 15 \wedge n - 6 < 46$.

38 Which tabulation below is a tabulation of the solution set of the condition for the problem?

a {53, 54, 55, . . ., 66}
b {38, 39, 40, . . ., 51}
c {38}
d {38, 39, 40, . . ., 53}

39 Which sentence below is the correct answer to the problem?

a Mr. Smith is 38 years old.
b Mr. Smith is from 53 through 66 years old.
c Mr. Smith is from 38 through 53 years old.
d Mr. Smith is from 38 through 51 years old.

Answers: 37-c, 38-b, 39-d

Use the following problem for exercises 40 and 41:

What numbers less than 110 can be subtracted from 227 so that each difference is less than 195?

40 ~~~ expresses a condition for the problem. U = N.

a $110 < y \wedge 227 - y < 195$.
b $y < 110 \wedge 227 - y < 195$.
c $y < 110 \wedge y - 227 > 195$.
d $y < 110 \wedge 227 - 195 > y$.

41 Which tabulation below is a tabulation of the solution set of the condition for the problem?

a {33, 34, 35, . . ., 109}
b {33, 34, 35, . . .}
c {0, 1, 2, . . ., 109}
d {111, 112, 113, . . .}

Answers: 40-b, 41-a

Test on unit 3

Conditions in two variables

Exercises 1-12 If a sentence expresses a true statement, write "T." Write "F" if the sentence expresses a false statement.

1 If A = {1, 2}, then {(1, 1), (2, 1), (1, 2), (2, 2)} is a tabulation of the Cartesian set A × A.

2 The set {(2, 3)} has exactly one member.

3 If U = N × N, there is exactly one point in the graph of $\{(x, y) | x = 2\}$.

4 The members of a Cartesian set are ordered pairs.

5 (5, 10) is the same ordered pair as (10, 5).

6 Each solution of $2x + y > 12$ is an ordered pair. U = N × N.

7 If A has 4 members, then A × A has 8 members.

8 If U = N × N, then (16, 35) is a member of $\{(x, y) | 2x + 3 = y\}$.

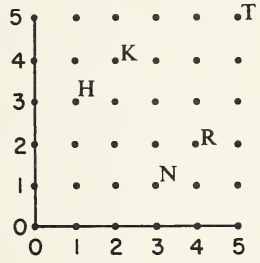
9 If U = N × N, then (3, 5) is a member of $\{(x, y) | x = 3 \wedge y = 10\}$.

10 The first component of each solution of $x < y \wedge x + y = 10$ is less than 5.

11 The two axes that are pictured in a graph are perpendicular.

12 A graph of a solution set is a subset of a graph of a universe.

Answers: 1-T, 2-T, 3-F, 4-T, 5-F, 6-T, 7-F, 8-T, 9-F, 10-T, 11-T, 12-T



Exercises 13-15 The letters H, K, N, R, and T label five points on the graph shown above. Use the graph in answering the questions.

13 What point is associated with (3, 1)?

a point H
b point K
c point N
d point R
e point T

14 What point is associated with (5, 5)?

a point H
b point K
c point N
d point R
e point T

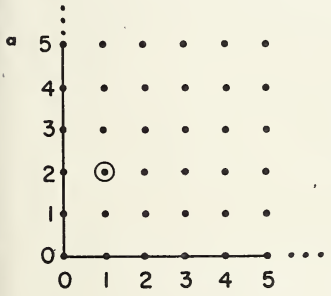
15 What point is associated with (2, 4)?

a point H
b point K
c point N
d point R
e point T

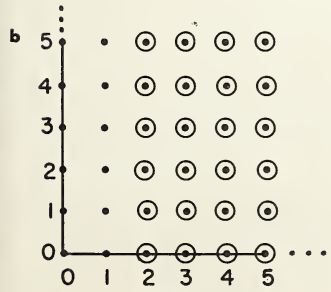
Answers: 13-c, 14-e, 15-b

Exercises 16-21 Five graphs are pictured below. For each exercise, select from these graphs the graph of the set named. The universe for (x, y) is $N \times N$. Each graph may be used once, more than once, or not at all.

a

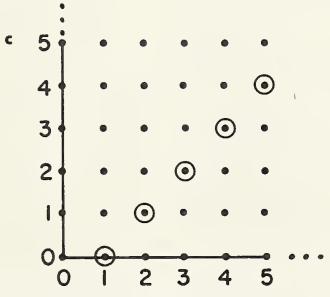


b



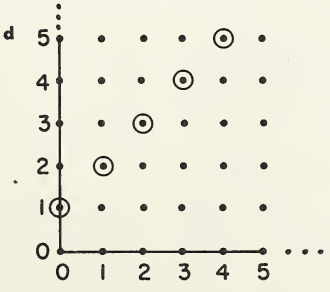
Incomplete graph

c



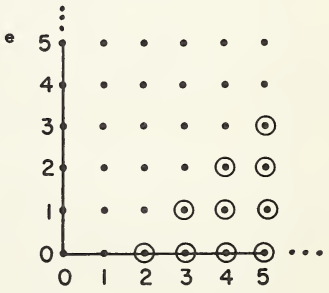
Incomplete graph

d



Incomplete graph

e



Incomplete graph

16 $\{(x, y) | y + 1 = x\}$ **18** $\{(x, y) | y + 1 < x\}$
17 $\{x, y | x > 1\}$ **19** $\{(x, y) | x + 1 = y\}$
20 $\{(x, y) | y - 1 = x\}$
21 $\{(x, y) | y + 4 = 6 \wedge x = 1\}$

Answers: 16-c, 17-b, 18-e, 19-d, 20-d, 21-a

Exercises 22-32 Four responses are listed for each exercise. Select the response that best answers the question in the exercise or that correctly completes each exercise.

22 What is the greatest number that you should use as a replacement for x to determine the solution set of $x + y = 9$? $U = N$.

a 7 **b** 8 **c** 9 **d** 10

23 The first components of the members of $R \times S$ are members of what set?

a $R \times S$ **b** $S \times R$ **c** S **d** R

24 $S = \{2, 4\}$. Which tabulation below is a tabulation of $S \times S$?

a {8}
b {(2, 2), (4, 4)}
c {(2, 2), (2, 4), (4, 2), (4, 4)}
d {(2, 4), (2, 2), (4, 4)}

25 The solution set of which condition expressed below is a finite set? $U = N \times N$.

a $2x + y > 30 \wedge x = 20$.
b $y < x + 10 \wedge y + x = 30$.
c $x - 28 = y$.
d $2x = 30$.

26 Which set named below does *not* contain (8, 12)? The universe for (x, y) is $N \times N$.

a $\{(x, y) | 3y - 28 = x\}$
b $\{(x, y) | y + 9 < 3x\}$
c $\{(x, y) | y - x = 4 \wedge x < 10\}$
d $\{(x, y) | 2x - y > 10 \wedge x < 10\}$

27 Which tabulation below is a tabulation of the solution set of $2s + 3 < 3t$ if the universe for $(s, t) = \{(0, 1), (0, 3), (2, 1), (2, 3)\}$?

a {(0, 3)}
b {(0, 3), (2, 3)}
c {(0, 2), (0, 3)}
d {(1, 3), (0, 2)}

28 Suppose that you are to make a chart to help find the solutions of $x + y < 8$. Which numerals should you use in the first row of your chart? $U = N \times N$.

a 0, 1, 2, 3, 4, 5, 6, 7
b 5, 6, 7, 8
c 0, 1, 2, 3, 4, 5, 6, 7, 8
d 1, 2, 3, 4

29 In a certain Cartesian set $A \times B$ there are just four members. (1, 7), (7, 5), and (7, 7) are in $A \times B$. What other ordered pair is in $A \times B$?

a (1, 5) **b** (5, 7) **c** (5, 1) **d** (5, 5)

30 What is the solution for the following problem?

The sum of Tom's age and Jack's age is 18. In four years, Tom will be as old as Jack is now. How old is each boy now? $U = C \times C$.

a Tom is 7 years old, and Jack is 11 years old.
b Tom is 14 years old, and Jack is 18 years old.
c Tom is 4 years old, and Jack is 14 years old.
d Tom is 11 years old, and Jack is 7 years old.

31 If you use x as a variable for the first number, what is a condition for the following problem?

When a first number is subtracted from a second number, the difference is 98. The product of 2 and the first number is 34. What are the numbers? $U = C \times C$.

a $x - y = 98 \wedge 2y = 34$.
b $y - x = 98 \wedge 2y = 34$.
c $x - y = 98 \wedge 2x = 34$.
d $y - x = 98 \wedge 2x = 34$.

32 The solution set of $x + 1 < 5 \wedge x + y = 3$ is ~~~~. $U = N \times N$.

a {0, 1, 2, 3}
b {(1, 2), (2, 1)}
c {(0, 3), (1, 2), (2, 1), (3, 0)}
d {(1, 2)}

Answers: 22-c, 23-d, 24-c, 25-d, 26-d, 27-b, 28-a, 29-a, 30-a, 31-d, 32-c

Exercises 33-35 Five conditions are expressed on page 259. For each exercise, select a condition for the problem. Each condition may be used once, several times, or not at all. $U = N \times N$.

a $x+y=7 \wedge x-y=5$.
b $x+y=7 \wedge 5-y=x$.
c $x+y=7 \wedge x+5=y$.
d $x+y=7$.
e $7+x=y$.

33 Together, Bill and Tom scored 7 points during a basketball game. How many points could each boy have scored?

34 The sum of two numbers is 7. The result of subtracting the second number from the first number is 5. What are the two numbers?

35 Jack had 7 coins. Then he received more coins. How many coins could he have received? How many coins could he have in all?

Answers: 33-d, 34-a, 35-e

Exercises 36-39 Five open sentences are listed below. For each exercise, select the sentence that expresses a condition for the problem. Each condition may be used once, several times, or not at all. $U = N \times N$.

a $x<5 \wedge x+7>y$.
b $x>7 \wedge x-5<5$.
c $x<5 \wedge x+y=7$.
d $x<5 \wedge 7+x=y$.
e $x+y<7 \wedge x<5$.

36 Mrs. Murphy bought fewer than five pounds of apples. Altogether she bought 7 pounds of apples and oranges. How many pounds of apples and how many pounds of oranges could she have bought?

37 Altogether, Mrs. Jones baked fewer than 7 pies and cakes. She baked fewer than 5 pies. How many pies and how many cakes could Mrs. Jones have baked?

38 Mary made fewer than five posters. If she makes seven more posters she will have made more posters than Joan made. How many posters could Mary have made, and how many posters could Joan have made?

39 The first of two numbers is less than 5. The result of adding the first number to 7 is the second number. What can the numbers be?

Answers: 36-c, 37-e, 38-a, 39-d

Test on unit 4

Conditions involving rate pairs

Exercises 1-11 Write "T" for each sentence that expresses a true statement. Write "F" for each sentence that expresses a false statement.

1 Ordered pairs of numbers can be used to represent rate situations.

2 (5, 3) and (12, 20) are equivalent ordered pairs.

3 The solution of $3/5 \sim x/35$, is also the solution of $3/5 \sim 35/x$. $U = C$.

4 $\{(5, 1), (4, 1), (3, 1), (2, 1)\}$ is a proportional relation.

5 A per cent is a rate pair.

6 $\{2/81, \ldots\} = \{6/243, \ldots\}$.

7 Any ordered pair that is equivalent to 26/39 is also equivalent to 8/12.

8 $\{7/13, \ldots\} = \{13/7, \ldots\}$.

9 $\{21/53, \ldots\}$ is an infinite set.

10 Forty-five miles per hour is the same rate as ninety miles in two hours.

11 If $U = C \times C$, $\{(x, y) | x/y \sim 2/3 \wedge x > y\}$ is the empty set.

Answers: 1-T, 2-F, 3-F, 4-F, 5-T, 6-T, 7-T, 8-F, 9-T, 10-T, 11-T

Exercises 12-17 Five sets are tabulated below. For each exercise, select from the list the tabulation of the set described. You may use a response once, more than once, or not at all. The universe for each variable is C.

a {35} b {45} c {55} d {65} e {75}

12 $13/5 \sim 117/m$.
13 s is 15% of 500.
14 $7/y \sim 19/95$.
15 $5/7 \sim n/91$.
16 $135/6 \sim k/2$.
17 13 is 20% of r.

Answers: 12-b, 13-e, 14-a, 15-d, 16-b, 17-d

Exercises 18-23 Five rate pairs are expressed below. For each exercise, select from this list a rate pair that is a member of the solution set of the condition. The universe for $(x, y) = N \times N$.

a 6/9 **b** 9/15 **c** 12/18 **d** 18/24 **e** 21/35

18 $x/y \sim 6/8 \wedge y - 6 = x$.

19 $10/15 \sim x/y \wedge y > x + 5$.

20 $2/3 \sim x/y \wedge x + 3 = y$.

21 $x + 10 < y \wedge x/y \sim 3/5$.

22 $x/y \sim 6/10 \wedge y - x = 6$.

23 $y/x \sim 15/10 \wedge y < x + 4$.

Answers: 18-d, 19-c, 20-a, 21-e, 22-b, 23-a

Problems 24-29 For each problem, select from the list below a sentence that expresses a condition for the problem. You may use a response once, more than once, or not at all. U = C.

a $40/100 \sim x/700 \wedge 700 - x = y$.

b $40/100 \sim x/700 \wedge 700 + x = y$.

c $40/100 \sim x/700$.

d $70/700 \sim x/100$.

e $40/700 \sim 100/x$.

24 Mrs. Jones bought furniture that usually sold for \$700. She received a 40% discount. The discount amounted to how many dollars?

25 Mr. Young, a merchant, buys goods for 40% of the price he sells them for. If he sells certain goods for \$700, what is the difference of the selling price and the price at which he bought the goods?

26 Mr. Jackson received a commission of \$70 for selling \$700 worth of merchandise. What rate of commission did Mr. Jackson receive?

27 For each \$700 that Mr. Thomas earns, he saves \$40. How many dollars must he earn to save \$100?

28 Mr. Avery wanted to buy a used car for \$700. When he bought a car, he paid 40% more than he had planned. How much did he pay?

29 Seventy is what per cent of seven hundred?

 Answers: 24-c, 25-a, 26-d, 27-e, 28-b, 29-d

Exercises 30-41 From the four responses below each exercise, select the response that best answers the question or that correctly completes the exercise.

30 To show that 3/6 and 5/10 are equivalent, what computation should you use?

a Multiply each component of 3/6 by the same counting number to obtain 5/10.

b Divide each component of 5/10 by the same counting number to obtain 3/6.

c Multiply 3 and 10, and multiply 5 and 6.

d Multiply 3 and 5, and multiply 6 and 10.

31 On a test, Bob correctly answered 17 of 20 problems. On another test, Jack correctly answered 27 of 30 problems. Which of the following best indicates which boy did better?

a Bob missed 3 problems, and Jack missed 3 problems.

b Jack correctly answered 10 problems more than Bob did.

c Bob correctly answered 17% of the problems on the test that he took, and Jack correctly answered 27% of the problems on the test that he took.

d Bob correctly answered 85% of the problems on the test that he took, and Jack correctly answered 90% of the problems on the test that he took.

32 Which of the following is *not* an explanation of why 6/8 and 12/16 are equivalent?

a $6(16) = 12(8)$.

b If each component of 6/8 is multiplied by 2, 12/16 is obtained.

c $6 < 8$ and $12 < 16$.

d $6/8 \sim 3/4$ and $12/16 \sim 3/4$.

33 Ann typed 315 words in 7 minutes. Which condition expressed below should you use to find how many words Ann typed per minute? The universe for the variable is C.

a $x/1 \sim 315/7$.

b $x/1 \sim 300/60$.

c $x/315 \sim 7/1$.

d $2205/5 \sim 1101/x$.

34 Monroe School has thirty times as many students as teachers. There are 900 students. Which condition should you use to find how many teachers there are at Monroe School? U = C.

a $1/30 \sim 900/y$.
c $30/1 \sim y/900$.
b $30/1 \sim 900/y$.
d $30/900 \sim y/1$.

35 To find the solution of $175/200 \sim 252/n$, you should first find the solution of ~~~. U = C.

a $175 + n = 200 + 252$.
c $175(200) = 252n$.
b $200n = 175(252)$.
d $175n = 252(200)$.

36 When a loan is obtained from a bank, the charge for the use of the money is called ~~~.

a principal
c commission
b discount
d interest

37 Mary wants to buy a phonograph that sells for $40. She has already saved 30% of the amount she needs. Which condition should you use to find how many more dollars she must save to buy the phonograph? U = C × C.

a $x/40 \sim 30/100 \wedge 40 + x = y$.
b $40/x \sim 30/100 \wedge y - x = 40$.
c $x/40 \sim 30/100 \wedge 40 + y = x$.
d $x/40 \sim 30/100 \wedge x + y = 40$.

38 A certain rate pair is equivalent to 3/5 and has a second component that is 16 more than the first component. Which condition should you use to find the rate pair? U = C × C.

a $x/y \sim 3/5 \wedge x + y = 16$.
b $x/y \sim 3/5 \wedge y - x = 16$.
c $x/y \sim 3/5 \wedge 16 > x - y$.
d $x/y \sim 3/5 \wedge y + 16 = x$.

39 When the selling price of an article is reduced, the amount of reduction is called ~~~.

a interest
c commission
b discount
d principal

40 ~~~ expresses the condition "18 is 36% of what number?"

a $1/8 \sim 36/x$.
c $36/18 \sim x/100$.
b $18/x \sim 36/100$.
d $18/36 \sim x/100$.

41 The fee paid to a salesman for his services in selling merchandise is called ~~~.

a commission
c bonus
b take-home pay
d dividend

Answers: 30-c, 31-d, 32-c, 33-a, 34-b, 35-d, 36-d, 37-d, 38-b, 39-b, 40-b, 41-a

Test on unit 5

Numeration systems

Exercises 1-17 If a sentence expresses a true statement, write "T." Write "F" if the sentence expresses a false statement.

1 A code system has the property of addition.
2 XXIV = 26.
3 $51_{\text{seven}} = 36$.
4 124_{five} is a numeral in a numeration system that has the property of place value.
5 $3^4 = 4^3$.
6 The numeration system used by the Babylonians had the property of place value.
7 The digit 8 in the decimal numeral 4832 is associated with the third power of ten.
8 $2 \times 10^6 = 2{,}000{,}000$.
9 All of the digits 0, 1, 2, 3, 4, 5, 6 are used in a base-six numeration system that has place value.
10 In scientific notation, $2000 = 20 \times 10^2$.
11 The fourth power of ten is ten thousand.
12 The numeral 20_{five} expresses the number forty.
13 A tally numeration system has the property of repetition.
14 The base of any grouping system must be a counting number greater than 1.
15 The base of 7^5 is 5.
16 The Egyptian numeration system had the property of place value.

17 The number 1 is a factor of every counting number.

Answers: 1-F, 2-F, 3-T, 4-T, 5-F, 6-T, 7-F, 8-T, 9-F, 10-F, 11-T, 12-F, 13-T, 14-T, 15-F, 16-F, 17-T

Exercises 18-23 Five numerals are listed below. For each exercise, select from the list the decimal numeral that expresses the number named. Each numeral listed as a response may be used once, more than once, or not at all.

a 35 **b** 65 **c** 85 **d** 105 **e** 155

18 71_{twelve} **20** 2102_{three} **22** 182_{nine}

19 43_{eight} **21** 253_{six} **23** 100011_{two}

Answers: 18-c, 19-a, 20-b, 21-d, 22-e, 23-a

Exercises 24-40 For each of exercises 24 through 27, four responses are listed below the exercise. Select the response that best answers the question in the exercise or that correctly completes the exercise.

24 The exponent of 4^3 is ~~~~.

a 1 **b** 3 **c** 4 **d** 12

25 Which number has eight as a factor?

a 4 **b** 8 **c** 38 **d** 108

26 In a base-four grouping system that does not have place value, there must be a special symbol for the number ~~~~.

a 2 **b** 8 **c** 10 **d** 16

27 What number does the base-five numeral below express?

a 13 **b** 203 **c** 493 **d** 3433

28 $6(10^3) + 4(10^2) + 0(10^1) + 7$ is the expanded form of ~~~~.

a 17 **b** 647 **c** 6407 **d** 6342017

29 Which digit in the numeral 8296 is associated with the second power of the base?

a 2 **b** 6 **c** 8 **d** 9

30 Which numeral expresses the number one hundred million two hundred sixty-five thousand thirty?

a 100,200,065,030 **c** 100,260,530

b 100,260,005,030 **d** 100,265,030

31 The base-six numeral that expresses the sum of 42_{six} and 35_{six} is ~~~~.

a 77_{six} **b** 121_{six} **c** 115_{six} **d** 1111_{six}

32 In the numeral 256,974,368,175, the digits ~~~~ form the billions' period.

a 256 **b** 974 **c** 368 **d** 175

33 If $3 \times 4 = 10$ expresses a true statement, the numerals 3, 4, and 10 must be written in base ~~~~.

a five **b** six **c** ten **d** twelve

34 What is the base of a place-value numeration system that uses exactly eight different digits?

a nine **b** eight **c** seven **d** six

35 The number seven million is expressed in scientific notation as ~~~~.

a 7×10^6 **b** 7×10^7 **c** 7×10^8 **d** 7×10^9

36 If the sentence $33 + 14 = 102$ expresses a true statement, the numerals 33, 14, and 102 are written in base ~~~~.

a three **b** four **c** five **d** seven

37 The digit 3 in the numeral 2356_{seven} expresses ~~~~.

a 21 **b** 147 **c** 300 **d** 2100

38 The numeral that expresses the number one thousand nine hundred forty-six is ~~~~.

a MCMXLVI **c** MDCDXLVI

b MCMLXIV **d** MDCCCCLXVI

39 The numeral $\mathcal{X}2\mathcal{E}_{twelve}$ expresses the same number as the base-ten numeral ~~~~.

a 1596 **b** 1475 **c** 133 **d** 23

40 770_{eight} is how many times as great as 77_{eight}?

a two **b** four **c** eight **d** ten

Answers: 24-b, 25-b, 26-d, 27-c, 28-c, 29-a, 30-d, 31-b, 32-a, 33-d, 34-b, 35-a, 36-c, 37-b, 38-a, 39-b, 40-c

Test on unit 6

The natural-number system

Exercises 1-15 If a sentence expresses a true statement, write "T." Write "F" if the sentence expresses a false statement.

1 {(1, 1), (2, 2), (3, 3)} can be put in one-to-one correspondence with {1, 2, 3}.

2 The number 24 is an example of a composite number.

3 Each natural number has a successor.

4 $1 = n\{(1, 2)\}$.

5 {0, 1, 2} is equivalent to {3}.

6 The standard set of 8 is a proper subset of the standard set of 9.

7 Addition of natural numbers is a binary operation.

8 If A and B are finite sets, $n(A \times B) = n(B \times A)$.

9 The quotient of two natural numbers is always a natural number.

10 57 is a prime number.

11 Each natural number has just one standard name.

12 The greatest common factor of two counting numbers must be less than each of the two numbers.

13 A finite set and an infinite set can be put in one-to-one correspondence.

14 The empty set does not have a proper subset.

15 The set of natural numbers is closed under subtraction.

Answers: 1-T, 2-T, 3-T, 4-T, 5-F, 6-T, 7-T, 8-T, 9-F, 10-F, 11-T, 12-F, 13-F, 14-T, 15-F

Exercises 16-21 The names of five properties of operations on natural numbers are listed in the next column. For each exercise, select the name of the property that explains why the sentence expresses a true statement. You may use a response once, more than once, or not at all.

a Distributive property of multiplication over addition
b Associative property of addition
c Commutative property of addition
d Associative property of multiplication
e Commutative property of multiplication

16 $1 + (2 + 3) = (1 + 2) + 3$.
17 $5(9 + 2) = (9 + 2)5$.
18 $(45 + 31) + 68 = 45 + (31 + 68)$.
19 $2(9 \cdot 3) = (2 \cdot 9)3$.
20 $(4 - 1)(6 \div 3) = (6 \div 3)(4 - 1)$.
21 $(35 + 18)92 = (18 + 35)92$.

Answers: 16-b, 17-e, 18-b, 19-d, 20-e, 21-c

Exercises 22-25 Five standard names of natural numbers are listed below. For each exercise, select from the list the standard name of the number named by the expression. You may use a response once, more than once, or not at all.

a 2 **b** 16 **c** 20 **d** 27 **e** 60

22 $12 \cdot 2 + 3$
23 $3 \cdot 2 + 2 \cdot 5$
24 $2 \cdot 3^2 + 3^2 \cdot 1$
25 $(15 \div 3) \cdot 4$

Answers: 22-d, 23-b, 24-d, 25-c

Exercises 26-40 Four responses are listed below each exercise. Select the response that best answers the question in the exercise or that correctly completes the exercise.

26 A set that is closed under multiplication is ~~~~.

a {1, 3, 5, . . ., 19}
b {2, 4, 6, . . ., 98}
c {0, 1, 2}
d {0, 1}

27 If $n(A) = 7$ and $n(B) = 32$, then the product 7×32 is associated with ~~~~.

a $A \cup B$
b $A \cap B$
c $A \times B$
d $A \times A$

28 If $n(R) = 10$ and $n(S) = 20$, and R and S are disjoint sets, what number is associated with $R \cup S$?

a 30 **b** 10 **c** 200 **d** 20

29 Which expression represents the successor of y? $U = N$.

a $y + 1$ **b** $y(1)$ **c** $y - 1$ **d** $y \div 1$

30 Which of the following sets is closed under addition?

a {2, 4, 6, . . ., 876}

b {5, 10, 15, . . .}

c {1, 3, 5, . . .}

d {0, 1}

31 Which one of the following powers of 2 is the greatest common factor of 24, 100, and 148?

a 2^4 **b** 2^3 **c** 2^2 **d** 2^1

32 The statement $(6 - 3) - 2 = 6 - (3 - 2)$ can be used to illustrate which of the following statements?

a Subtraction of natural numbers is not commutative.

b Subtraction of natural numbers is not associative.

c The set of natural numbers is not closed under subtraction.

d Subtraction of natural numbers does not distribute over addition.

33 The property of natural numbers that can be used to help find the solution of $48(n) = (132)48$ is the ——. $U = N$.

a commutative property of multiplication

b associative property of multiplication

c distributive property of multiplication over addition

d identity-element property of multiplication

34 Which of the following responses represents a prime number?

a the third power of three

b the greatest common factor of 12 and 18

c n{0, 1, 2, 3}

d $n(A \times B)$, when $A = \{0\}$ and $B = \{0, 1, 2\}$

35 The complete factorization of 84 is ——.

a 21×4 **c** $1 \times 2 \times 2 \times 3 \times 7$

b $2^2 \times 7 \times 3$ **d** $4 \times 3 \times 7$

36 The statement $117 > 29$ gives the same information as ——.

a $29 > 117$.

b $117 \neq 29$.

c $117(29) = 29(117)$.

d $29 < 117$.

37 $30 - 12 = 18$ also means ——.

a $30 - 2 = 28$. **c** $30 = 12 + 18$.

b $30 = 22 + 8$. **d** $30 - 8 = 22$.

38 Which of the following is an example of twin primes?

a 9, 10 **b** 11, 13 **c** 13, 15 **d** 21, 23

39 Which of the following quotients cannot be found?

a $24 \div 0$ **c** $85 \div 17$

b $0 \div 24$ **d** $182 \div 13$

40 The number that is associated with {0, 1, 2, 3, 4} is the successor of ——.

a 5 **b** 6 **c** 4 **d** 7

Answers: 26-d, 27-c, 28-a, 29-a, 30-b, 31-c, 32-b, 33-a, 34-d, 35-b, 36-d, 37-c, 38-b, 39-a, 40-c

---

Test on unit 7

The rational numbers of arithmetic

Exercises 1-20 If a sentence expresses a true statement, write "T." Write "F" if the sentence expresses a false statement.

1 $\frac{18}{3}$ is a basic fraction.

2 The fraction $\frac{2}{5}$ can be obtained from $\frac{14}{35}$ by using the reduction property of fractions.

3 $\{\frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \ldots\}$ is closed under addition.

4 $\frac{2}{3} > \frac{29}{43}$.

5 The fraction $\frac{5}{3}$ is equivalent to the fraction $\frac{5+2}{3+2}$.

6 For each replacement of r and s, a true statement is obtained from $\frac{0}{r} = \frac{0}{s}$. The universe for r and s is C.

7 329 is a multiple of 7.

8 For each replacement of a and b, a true statement is obtained from $\frac{a}{3} + \frac{b}{3} = \frac{a+b}{6}$. The universe for a and b is N.

9 15×24 is the mate of $\frac{15}{1} \times \frac{24}{1}$.

10 The difference of any two rational numbers of arithmetic is a rational number of arithmetic.

11 $\frac{25}{3} + \frac{5}{10} = \frac{250+15}{30}$.

12 The identity element for addition of rational numbers is 1.

13 $\frac{1}{2} \div \frac{1}{4} = y$ has the same solution set as $\frac{1}{2}(4) = y$. $U = R_a$.

14 The reciprocal of $\frac{6}{13}$ is $\frac{1}{\frac{13}{6}}$.

15 Division of rational numbers has the associative property.

16 $3.46 \div 2.6 = 34.6 \div 26$.

17 The expanded form of .253 is $\frac{2}{10} + \frac{5}{100} + \frac{3}{1000}$.

18 $1/10 \sim 1/100$.

19 $\{y | y = \frac{1}{8} \div \frac{5}{6}\} = \{\frac{3}{20}\}$ $U = Z$.

20 $.5\overline{1} > .5$.

Answers: 1-F, 2-T, 3-T, 4-F, 5-F, 6-T, 7-T, 8-F, 9-T, 10-F, 11-T, 12-F, 13-T, 14-F, 15-F, 16-T, 17-T, 18-F, 19-T, 20-T

Exercises 21-25 The names of five properties of multiplication of rational numbers of arithmetic are listed below. Examples of properties are given in the exercises. For each exercise, select the name of the property illustrated. You may use a response once, more than once, or not at all.

- **a** Closure property
- **b** Commutative property
- **c** Associative property
- **d** Identity-element property
- **e** Zero property

21 $1 \cdot \frac{12}{5} = \frac{12}{5}$.

22 $0 \cdot \frac{7}{8} = 0$.

23 $\frac{4}{9}(\frac{8}{18} \cdot \frac{5}{6}) = (\frac{4}{9} \cdot \frac{8}{18})\frac{5}{6}$.

24 $\frac{11}{15} \cdot \frac{32}{49}$ is a member of R_a.

25 $(\frac{1}{2} - \frac{2}{4})\frac{3}{7} = \frac{0}{14}$.

Answers: 21-d, 22-e, 23-c, 24-a, 25-e

Exercises 26-42 A choice of four responses is given for each exercise. Select the correct response and write its label on the response line.

26 Which sentence below shows that $\frac{7}{6} < \frac{9}{7}$?

- **a** $7 < 9$ and $6 < 7$.
- **c** $7(7) < 6(9)$.
- **b** $7(6) < 9(6)$.
- **d** $7(6) < 9(7)$.

27 What property of rational numbers should you use to show that, if $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ is a true statement, then $\frac{5}{10} + \frac{2}{6} = \frac{5}{6}$ is also a true statement?

- **a** The definition of sum
- **b** The well-defined property of addition
- **c** The closure property of addition
- **d** The commutative property of addition

28 Which sentence below expresses the commutative property of addition of rational numbers? $U = R_a$.

- **a** $x + y$ is a member of R_a.
- **b** $x + (y + z) = (x + y) + z$.
- **c** $x + 0 = x$.
- **d** $x + y = y + x$.

29 The solution set of $\frac{8}{5} - x = \frac{2}{3}$ is the same as the solution set of which condition expressed below?

- **a** $x = \frac{2}{3} + \frac{8}{5}$.
- **c** $x = \frac{8}{5} - \frac{2}{3}$.
- **b** $x - \frac{8}{5} = \frac{2}{3}$.
- **d** $x - \frac{2}{3} = \frac{8}{5}$.

30 Which numeral given below is another name for $\frac{51}{6}$?

- **a** $5\frac{1}{6}$
- **b** $9\frac{3}{6}$
- **c** $8\frac{5}{6}$
- **d** $8\frac{1}{2}$

31 Which condition expressed below is a property of rational numbers? $U = R_a$.

- **a** $(0)y = y$.
- **c** $y(1) = 1$.
- **b** $y \div 1 = y$.
- **d** $y + y = 0$.

32 Which digit in 382.5746 is associated with $\frac{1}{10^2}$?

a 3 b 5 c 7 d 8

33 What is the solution of $3\frac{1}{4}/x \sim 13/1\frac{3}{4}$? U = Z.

a $\frac{169}{7}$ b $\frac{4}{7}$ c $\frac{1183}{16}$ d $\frac{7}{16}$

34 What number is $16\frac{2}{3}\%$ of 300?

a 180 b 50 c 18 d 60

35 280 is $87\frac{1}{2}\%$ of what number?

a $192\frac{1}{2}$ b 245 c 320 d $367\frac{1}{2}$

36 84 is what fraction of 36?

a $\frac{7}{3}$ b $\frac{2}{1}$ c $\frac{1}{2}$ d $\frac{3}{7}$

37 What is the greatest common divisor of 12 and 96?

a 4 b 6 c 12 d 24

38 How should the numeral .264 be read?

a "Two hundred and sixty-four thousandths"

b "Point two and sixty-four thousandths"

c "Point two sixty-four thousandths"

d "Two hundred sixty-four thousandths"

39 Last year Jane was paid 40¢ per hour for baby-sitting. This year she is paid 50¢ per hour. What is the per cent of increase?

a 20% b 25% c 80% d $11\frac{1}{9}\%$

40 What is the solution of $\frac{2}{5}/1 \sim x/40$?

a 200 b 8 c 16 d 100

41 What computation should you use to find the solution of $\frac{1}{4}x = 22$?

a Multiply 22 by 4?

b Multiply 22 by $\frac{1}{4}$

c Divide 22 by 4

d Divide $\frac{1}{4}$ by 22

42 Which is the decimal for $\frac{5}{6}$?

a $8.\overline{3}$ b $.08\overline{3}$ c $.\overline{83}$ d $.8\overline{3}$

Answers: 26-c, 27-b, 28-d, 29-c, 30-d, 31-b, 32-c, 33-d, 34-b, 35-c, 36-a, 37-c, 38-d, 39-b, 40-c, 41-a, 42-d

---

End-of-book test

Exercises 1-7 Five responses are listed below. For each exercise, select from the list the response that best completes the exercise. You may use a response once, more than once, or not at all.

a $\{(1, 0), (2, 0)\}$ d $\{(1, 1), (2, 2)\}$

b $\{(1, 2)\}$ e $\{\ \}$

c $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$

1 $\{(1, 1), (1, 2), (2, 1), (2, 2)\} \cup \{(1, 1), (2, 2)\} =$ ———.

2 If G = {1, 2} and H = {0}, then G × H ———.

3 $\{(1, 1), (1, 2), (2, 1), (2, 2)\} \cap \{(1, 1), (2, 2)\} =$ ———.

4 $\{(1, 0), (2, 0)\} \cap \{\ \} =$ ———.

5 If D = {1, 2} and E = { }, then D × E = ———.

6 If A = {1, 2}, then A × A = ———.

7 { } ∪ $\{(1, 0), (2, 0)\}$ = ———.

Answers: 1-c, 2-a, 3-d, 4-e, 5-e, 6-c, 7-a

Exercises 8-24 Write "T" for each sentence that expresses a true statement. Write "F" for each sentence that expresses a false statement.

8 If points J and K are contained in plane ABC, then $\overleftrightarrow{JK}$ is a subset of plane ABC.

9 There are exactly two planes that contain three noncollinear points.

10 {(1, 2)} contains exactly one member.

11 A circle has infinitely many diameters.

12 If $\overline{AB} < \overline{CD}$ and $\overline{CD} < \overline{EF}$, then $\overline{AB} < \overline{EF}$.

13 If ℓ_1 and ℓ_2 are skew lines, then $\ell_1 \cap \ell_2 = \{\ \}$.

14 A variable in a condition may be replaced only by the members of the solution set of the condition.

15 $\{(x, y) | 2/3 \sim x/y\}$ contains exactly one member. U = C × C.

16 The intersection of sets A and B is a subset of the union of sets A and B.

17 The successor of 8 is 7.

18 With respect to addition and multiplication, {6, 12, 18, . . .} forms a number system.

19 {1} is a proper subset of {0, 1}.

20 The intersection of the interior of ∠ABC and the exterior of ∠ABC is ∠ABC.

21 Each member of the non-zero rational numbers of arithmetic has exactly one reciprocal.

22 (1, 2) and (2, 3) are equivalent rate pairs.
23 The natural-number mate of $\frac{7}{7}$ is 1.
24 A rate pair that represents a comparison and that has a second component of 100 is a per cent.

Answers: 8-T, 9-F, 10-T, 11-T, 12-T, 13-T, 14-F, 15-F, 16-T, 17-F, 18-T, 19-T, 20-F, 21-T, 22-F, 23-T, 24-T

Exercises 25-28 The names of five properties of R_a are listed below. The condition expressed in each exercise is a property. Choose the name of the property from the list. The universe for x, y, and z is R_a.

a Commutative property of multiplication
b Associative property of multiplication
c Closure property of multiplication
d Identity-element property of multiplication
e Zero property of multiplication

25 $1 \cdot y = y$.
26 $(xy)z = x(yz)$
27 $0 \cdot z = 0$.
28 $xy = yx$.

Answers: 25-d, 26-b, 27-e, 28-a

Exercises 29-47 Four responses are listed below each exercise. Select the response that best completes the exercise.

29 An example of a sentence that expresses a false statement is ~~~.
a $3 + 7 = 10$.
b $4 + 9 > 12$.
c $2 + 3 < 9$.
d $18 - 13 \neq 5$.

30 An example of an infinite set is ~~~.
a {0, 4, 8, . . .}
b { }
c {1, 2, 3, . . ., 100}
d $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots; \frac{1}{512}\}$

31 {4, 5, 6} and {40, 50, 60} are ~~~ sets.
a intersecting
b congruent
c equivalent
d equal

32 Points contained in the same line are ~~~ points.
a collinear
b noncollinear
c congruent
d intersecting

33 One ~~~ of {0, 7, 14, 21, 28, 35, 42} is 35.
a tabulation
b intersection
c member
d coördinate

34 A ~~~ separates a plane into half-planes.
a point
b line
c segment
d triangle

35 A rate pair equivalent to 1/400 is ~~~.
a 4/100
b 2.5%
c 25/100
d $\frac{1}{4}$%

36 The decimal numeration system does *not* have the property of ~~~.
a addition
b subtraction
c repetition
d place value

37 Another name for 400,000 is ~~~.
a 4×10^5
b 4×10^6
c 4×10^3
d 4×10^4

38 When a number is expressed as the product of primes, the expression is the ~~~ of the number.
a complete factorization
b exponential notation
c scientific notation
d expanded form

39 ~~~ gives the same information as $9 > 0$.
a $0 > 9$
b $5 > 4$
c $9 < 0$
d $0 < 9$

40 The expanded form of the numeral .073 is
a $0(\frac{1}{10^1}) + 7(\frac{1}{10^2}) + 3(\frac{1}{10^3})$
b $0(1) + 7(\frac{1}{10^1}) + 3(\frac{1}{10^2})$
c $0(\frac{1}{10}) + 7(\frac{1}{10^1}) + 3(\frac{1}{10^3})$
d $7(\frac{1}{10^1}) + 3(\frac{1}{10^2})$

41 A solution of $x - \frac{3}{5} > \frac{7}{10}$ is ~~~. $U = R_a$.
a $\frac{4}{10}$
b $\frac{10}{10}$
c $\frac{13}{10}$
d $\frac{14}{10}$

42 The solution set of $2.5 - x = .18$ is the same as the solution set of ~~~. $U = R_a$.
a $x - 2.5 = .18$
b $2.5 - .18 = x$
c $.18 - x = 2.5$
d $250 - x = 18$

43 The solution set of $r < 5(1.7) \wedge 2\frac{1}{2} + r > 6$ is ~~~.
a $\{r | r < 8\frac{1}{2}\}$
b $\{r | r > 3.5 \wedge r < 8.5\}$
c $\{r | r < 8.5\}$
d $\{r | r < 3.5 \wedge r > 8.5\}$

44 $U = Z$. $\{x | x/5 \sim 2\frac{1}{2}/\frac{1}{10}\}$ is equal to ~~~.
a $\{12\frac{1}{2}\}$
b {125}
c {5}
d {12}

45 ~~~ is the set of all possible factors of 28.
a {2, 4, 7, 14}
b {7, 14, 21, 28}
c { 1, 2, 4, 7, 14}
d {1, 2, 4, 7, 14, 28}

46 ~~~ is not a number system with respect to addition and multiplication.

a {48, 50, 52, . . .} c {5, 10, 15, . . .}
b {13, 15, 17, . . .} d {11, 22, 33, . . .}

47 If $\triangle CDE \cong \triangle FGH$, then segment ~~~ is congruent to $\overline{FH}$.

a CD b FG c DE d CE

Answers: 29-d, 30-a, 31-c, 32-a, 33-c, 34-b, 35-d, 36-b, 37-a, 38-a, 39-d, 40-a, 41-d, 42-b, 43-b, 44-b, 45-d, 46-b, 47-d.

Mathematical properties

■ These mathematical properties are developed and discussed in *STM 1*. The properties are classified under broad, general headings to make it easy for you to refer to a specific property. The numeral that follows each property identifies the page in *STM 1* on which the property first appears.

Properties of the system of natural numbers

CLOSURE PROPERTY OF
ADDITION OF NATURAL NUMBERS

The universe for a *and* b *is N. For each replacement of* a *and* b, *there is a number* a + b *that is a member of N.* 259

CLOSURE PROPERTY OF
MULTIPLICATION OF NATURAL NUMBERS

The universe for a *and* b *is N. For each replacement of* a *and* b, *there is a number* a × b *that is a member of N.* 263

COMMUTATIVE PROPERTY OF
ADDITION OF NATURAL NUMBERS

The universe for a *and* b *is N. For each replacement of* a *and* b, *a true statement is obtained from* a + b = b + a. 265

COMMUTATIVE PROPERTY OF
MULTIPLICATION OF NATURAL NUMBERS

The universe for a *and* b *is N. For each replacement of* a *and* b, *a true statement is obtained from* a × b = b × a. 267

ASSOCIATIVE PROPERTY OF
ADDITION OF NATURAL NUMBERS

The universe for a, b, *and* c *is N. For each replacement of* a, b, *and* c, *a true statement is obtained from* (a + b) + c = a + (b + c). 270

ASSOCIATIVE PROPERTY OF
MULTIPLICATION OF NATURAL NUMBERS

The universe for a, b, *and* c *is N. For each replacement of* a, b, *and* c, *a true statement is obtained from* (ab)c = a(bc). 271

DISTRIBUTIVE PROPERTY OF
MULTIPLICATION OVER ADDITION
OF NATURAL NUMBERS

The universe for a, b, *and* c *is N. For each replacement of* a, b, *and* c, *a true statement is obtained from* a(b + c) = (ab) + (ac). 275

Other properties of the set of natural numbers

IDENTITY-ELEMENT PROPERTY OF
ADDITION OF NATURAL NUMBERS

The universe for a *is N. For each replacemen*

of a, *a true statement is obtained from* $0 + a = a$. 289

IDENTITY-ELEMENT PROPERTY OF MULTIPLICATION OF NATURAL NUMBERS

The universe for a *is N. For each replacement of the variable, a true statement is obtained from* $1 \cdot a = a$. 290

ZERO PROPERTY OF MULTIPLICATION OF NATURAL NUMBERS

The universe for a *is N. For each replacement of* a, *a true statement is obtained from* $0 \cdot a = 0$. 291

Properties of the system of the rational numbers of arithmetic

CLOSURE PROPERTY OF MULTIPLICATION OF RATIONAL NUMBERS OF ARITHMETIC

The universe for x *and* y *is* R_a. *For each replacement of* x *and* y, *there is a number* xy *that is a member of* R_a. 326

CLOSURE PROPERTY OF ADDITION OF RATIONAL NUMBERS OF ARITHMETIC

The universe for x *and* y *is* R_a. *For each replacement of* x *and* y, *there is a number* $x + y$ *that is a member of* R_a. 329

COMMUTATIVE PROPERTY OF MULTIPLICATION OF RATIONAL NUMBERS OF ARITHMETIC

The universe for x *and* y *is* R_a. *For each replacement of the variables*, $xy = yx$. 349

COMMUTATIVE PROPERTY OF ADDITION OF RATIONAL NUMBERS OF ARITHMETIC

The universe for x *and* y *is* R_a. *For each replacement of the variables*, $x + y = y + x$. 350

ASSOCIATIVE PROPERTY OF MULTIPLICATION OF RATIONAL NUMBERS OF ARITHMETIC

The universe for x, y, *and* z *is* R_a. *For each replacement of the variables*, $(xy)z = x(yz)$. 352

ASSOCIATIVE PROPERTY OF ADDITION OF RATIONAL NUMBERS OF ARITHMETIC

The universe for x, y, *and* z *is* R_a. *For each replacement of the variables*, $(x + y) + z = x + (y + z)$. 353

DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION OF RATIONAL NUMBERS OF ARITHMETIC

The universe for x, y, *and* z *is* R_a. *For each replacement of the variables*, $x(y + z) = xy + xz$. 356

Other properties of the rational numbers of arithmetic

REDUCTION PROPERTY OF RATIONAL NUMBERS OF ARITHMETIC

The universe for a *is N. The universe for* b *and* k *is C. For each replacement of* a, b, *and* k, $\frac{a}{b} = \frac{ka}{kb}$. 317

SUM PROPERTY OF RATIONAL NUMBERS OF ARITHMETIC

The universe for a *and* c *is N. The universe for* b *and* d *is C. For each replacement of* a, b, c, *and* d, $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$. 331

DIFFERENCE PROPERTY OF
RATIONAL NUMBERS OF ARITHMETIC

The universe for a *and* c *is N. The universe for* b *and* d *is C. For each replacement of* a, b, c, *and* d, *if* ad *is greater than or equal to* bc, *then* $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$. 334

IDENTITY-ELEMENT PROPERTY OF
ADDITION OF RATIONAL NUMBERS
OF ARITHMETIC

The universe for x *is* R_a. *For each replacement of the variable,* $0 + x = x$. 361

IDENTITY-ELEMENT PROPERTY OF
MULTIPLICATION OF RATIONAL
NUMBERS OF ARITHMETIC

The universe for x *is* R_a. *For each replacement of the variable,* $1 \cdot x = x$. 362

ZERO PROPERTY OF
MULTIPLICATION OF RATIONAL
NUMBERS OF ARITHMETIC

The universe for x *is* R_a. *For each replacement of the variable,* $0 \cdot x = 0$. 362

RECIPROCAL PROPERTY OF
RATIONAL NUMBERS
OF ARITHMETIC

The universe for a *and* b *is C. For each replacement of* a *and* b, *the reciprocal of* $\frac{a}{b}$ *is* $\frac{b}{a}$. *The reciprocal of* $\frac{b}{a}$ *is* $\frac{a}{b}$. 364

The reciprocal property can be expressed in another way:

The universe for x *is the set of non-zero rational numbers. For each replacement of* x, *the reciprocal of* x *is* $1/x$. 369

QUOTIENT PROPERTY OF
RATIONAL NUMBERS
OF ARITHMETIC

The universe for a *is N. The universe for* b, c, *and* d *is C. For each replacement of* a, b, c, *and* d, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$. 366

If the reciprocal of the non-zero rational number x is expressed as $1/x$, the quotient property of rational numbers of arithmetic can be expressed in another way:

The universe for x *is* R_a. *The universe for* y *is the set of non-zero rational numbers of arithmetic. For each replacement of the variables,* $x \div y = x \cdot 1/y$. 369

DENSITY PROPERTY OF
THE SET OF RATIONAL NUMBERS
OF ARITHMETIC

The universe for x, y, *and* r *is* R_a. *For each replacement of* x *and* y, *if* $x < y$, *then there is a rational number* r *such that* $x < r \wedge r < y$. 389

Properties of equality of the rational numbers of arithmetic

WELL-DEFINED PROPERTY OF
MULTIPLICATION OF RATIONAL
NUMBERS OF ARITHMETIC

If one or both of the numbers used in forming the product of two rational numbers is replaced by an equal rational number, the product is still the same rational number. 326

WELL-DEFINED PROPERTY OF
ADDITION OF RATIONAL NUMBERS
OF ARITHMETIC

If the numbers used in forming the sum of two rational numbers are replaced by equal rational numbers, the sum is still the same rational number. 329

TRANSITIVE PROPERTY OF EQUALITY OF RATIONAL NUMBERS OF ARITHMETIC

The universe for x, y, *and* z *is* R_a. *For each replacement of* x, y, *and* z, *if* $x = y$ *and* $y = z$, *then* $x = z$. 348

Miscellaneous properties of numbers

UNIQUE FACTORIZATION PROPERTY OF COUNTING NUMBERS

Except for the order in which the factors occur, every counting number greater than 1 can be factored into primes in only one way. 301

REDUCTION PROPERTY OF FRACTIONS

If you multiply the numerator and the denominator of a given fraction by the same counting number, the fraction that you obtain is equivalent to the given fraction. If you divide the numerator and the denominator of a given fraction by a common factor, the fraction that you obtain is equivalent to the given fraction. 314

The same property also may be expressed as follows:

The universe for a *is N. The universe for* b *and* k *is C. For each replacement of the variables,* $\frac{a}{b} \sim \frac{ka}{kb}$. 314

Geometric properties relating points, lines, and planes

For any two different points there is exactly one line that contains both of these points. 33

If a line contains two points of a plane, the line is included in the plane. 34

A given line is included in infinitely many planes. 35

There is exactly one plane that includes a given line and contains a point not in the line. 35

There is exactly one plane that contains three noncollinear points. 35

If one point is between two other points, then the three points are collinear. 37

For every three collinear points, just one of the points is between the other two. 37

For any two points, there is at least one point between them. 37

The intersection of two parallel planes is the empty set. 99

The intersection of two planes that meet is a line. 100

The intersection of a line and a plane that are parallel is the empty set. 101

If a line and a plane intersect, their intersection either contains exactly one point or is the line. 101

The intersection of two parallel lines is the empty set. The intersection of two skew lines is the empty set. 101

If two lines intersect, their intersection contains exactly one point. 102

A simple closed curve separates the plane into two regions. 115

Geometric properties of segments

If $\overline{AB} \cong \overline{CD}$ *and* $\overline{CD} \cong \overline{EF}$ *are true statements, then* $\overline{AB} \cong \overline{EF}$ *is a true statement.* 39

This property can also be expressed in the shorter way shown below:

If $\overline{AB} \cong \overline{CD}$ *and* $\overline{CD} \cong \overline{EF}$, *then* $\overline{AB} \cong \overline{EF}$. 40

If $\overline{AB} < \overline{CD}$, *and* $\overline{CD} < \overline{EF}$, *then* $\overline{AB} < \overline{EF}$. 43

Bibliography

■ Two bibliographies, one of interest to teachers and one that should be helpful to students, are listed below and on the following two pages. See the Directory of Publishers in the back of this *Teaching Guide* for a list of the Canadian representatives of the publishers included in these bibliographies.

Of interest to teachers

Alexandroff, P. S. *An Introduction to the Theory of Groups*. London: Blackie and Son, Ltd., 1959.

Allendoerfer, C. B., and Oakley, C. O. *Fundamentals of Freshman Mathematics*. New York: McGraw-Hill Book Co., 1959.

Banks, J. Houston. *Elements of Mathematics*. Boston: Allyn and Bacon, Inc., 1961.

Birkhoff, G. D., and Beatley, Ralph. *Basic Geometry*. New York: Chelsea Publishing Co., 1941.

Birkhoff, Garrett, and Mac Lane, Saunders. *A Survey of Modern Algebra*. New York: Macmillan Co., 1953.

Boehm, George A. W. *The New World of Mathematics*. New York: Dial Press, 1959.

Breuer, Joseph. *Introduction to the Theory of Sets*. Translated by Howard F. Fehr. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1958.

Bruner, Jerome S. *The Process of Education*. Cambridge, Massachusetts: Harvard University Press, 1960.

Cameron, Edward A. *Algebra and Trigonometry*. New York: Holt, Rinehart and Winston, Inc., 1960.

Coxeter, H. S. M. *Introduction to Geometry*. New York: John Wiley and Sons, Inc., 1961.

Evenson, A. B. *Modern Mathematics*. Chicago: Scott, Foresman and Co., 1962.

Eves, H., and Newsom, C. V. *An Introduction to the Foundations and Fundamental Concepts of Mathematics*. New York: Rinehart and Co., 1958.

Goldberg, Samuel. *Probability, an Introduction*. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1960.

Halmos, Paul R. *Naive Set Theory*. Princeton, New Jersey: D. Van Nostrand Co., Inc., 1960.

Hilbert, David. *Foundations of Geometry*. La Salle, Illinois: Open Court Publishing Co., 1950.

Kelley, John L. *Introduction to Modern Algebra*. Princeton, New Jersey: D. Van Nostrand Co., Inc., 1960.

Kemeny, John G.; Snell, J. Laurie; and Thompson, Gerald L. *Introduction to Finite Mathematics*. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1957.

Landau, Edmund. *Foundations of Analysis*. New York: Chelsea Publishing Co., 1957.

May, K. O. *Elements of Modern Mathematics*. Reading, Massachusetts: Addison-Wesley Publishing Co., Inc., 1959.

Mosteller, Frederick; Rourke, Robert E. K.; and Thomas, George B., Jr. *Probability with Statistical Applications*. Reading, Massachusetts: Addison-Wesley Publishing Co., Inc., 1961.

National Council of Teachers of Mathematics. *The Growth of Mathematical Ideas: Grades K-12*. Twenty-Fourth Yearbook. Washington, D.C.: National Council of Teachers of Mathematics, 1959.

National Council of Teachers of Mathematics. *Insights into Modern Mathematics*. Twenty-Third Yearbook. Washington, D.C.: National Council of Teachers of Mathematics, 1957.

National Council of Teachers of Mathematics. *Instruction in Arithmetic*. Twenty-Fifth Yearbook. Washington, D.C.: National Council of Teachers of Mathematics, 1960.

National Council of Teachers of Mathematics.

The Revolution in School Mathematics. Washington, D.C.: National Council of Teachers of Mathematics, 1961.

Niven, Ivan. *Numbers: Rational and Irrational*. New York: Random House, 1961.

Polya, George. *How to Solve It*. New York: Doubleday and Co., 1957.

———. *Mathematical Discovery*. New York: John Wiley and Sons, Inc., 1962.

———. *Mathematics and Plausible Reasoning*. 2 vols. Princeton, New Jersey: Princeton University Press, 1954.

Report of the Commission on Mathematics. *Appendices*. College Entrance Examination Board, c/o Educational Testing Service, Box 592, Princeton, New Jersey. New York: College Entrance Examination Board, 1959.

Report of the Commission on Mathematics. *Program for college preparatory mathematics*. College Entrance Examination Board, c/o Educational Testing Service, Box 592, Princeton, New Jersey. New York: College Entrance Examination Board, 1959.

Sawyer, W. W. *A Concrete Approach to Abstract Algebra*. San Francisco: W. H. Freeman and Co., 1959.

Stabler, E. R. *An Introduction to Mathematical Thought*. Reading, Massachusetts: Addison-Wesley Publishing Co., Inc., 1953.

Stoll, Robert R. *Sets, Logic, and Axiomatic Theories*. San Francisco: W. H. Freeman and Co., 1961.

Suppes, Patrick. *Axiomatic Set Theory*. Princeton, New Jersey: D. Van Nostrand Co., Inc., 1960.

———. *Introduction to Logic*. Princeton, New Jersey: D. Van Nostrand Co., 1957.

Trimble, Harold C.; Hamilton, E. W.; and Silvey, I. M. *Mathematics for General Education*. 3d ed. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1963.

University of Chicago College Mathematics Staff. *Concepts and Structure of Mathematics*. Chicago: University of Chicago Press, 1954.

Van Der Waerden, B. L. *Science Awakening*. Distributed by Oxford University Press, Inc., New York. Groningen, Holland: P. Noordhoff, Ltd., 1954.

Weiss, Marie J., and Dubisch, Roy. *Higher Algebra for the Undergraduate*. New York: John Wiley and Sons, Inc., 1962.

Zehna, Peter W., and Johnson, Robert L. *Elements of Set Theory*. Boston: Allyn and Bacon, Inc., 1962.

Helpful to students

Abbott, Edwin A. *Flatland*. New York: Dover Publications, Inc., 1952.

Adler, Irving. *Mathematics*. New York: Golden Press, Inc., 1960.

Adler, Irving, and Adler, Ruth. *The New Mathematics*. New York: John Day and Co., 1958.

———. *Numbers Old and New*. New York: John Day and Co., 1960.

Andrews, F. Emerson. *Numbers, Please*. Boston: Little, Brown and Co., 1961.

Asimov, Isaac. *Realm of Measure*. Boston: Houghton, Mifflin Co., 1960.

Bakst, Aaron. *Mathematics, Its Magic and Mastery*. Princeton, New Jersey: D. Van Nostrand Co., Inc., 1952.

Ball, W. W. Rouse. *Mathematical Recreations and Essays*. New York: Macmillan Co., 1947.

Bell, E. T. *Men of Mathematics*. New York: Simon and Schuster, Inc., 1937.

Bendick, Jeanne, and Levin, Marcia O. *Take a Number*. New York: McGraw-Hill Book Co., 1961.

Dantzig, Tobias. *Number, the Language of Science*. New York: Doubleday and Co., Inc., 1956.

Davis, P. J. *The Lore of Large Numbers*. New York: Random House, 1961.

Freebury, H. A. *A History of Mathematics for Secondary Schools*. London: Cassell and Co., Ltd., 1958.

Gardner, Martin. *Mathematical Puzzles & Diversions*. New York: Simon and Schuster, Inc., 1961.

Gilles, William F. *The Magic and Oddities of Numbers*. New York: Vantage Press, Inc., 1953.

Glenn, William H., and Johnson, Donovan A. *Exploring Mathematics on Your Own*. New York: Doubleday and Co., Inc., 1961.

Heath, Royal V. *Mathemagic*. New York: Dover Publications, Inc., 1953.

Kline, M., and Crown, A. W. *The Language of Shapes, Books 1, 2, 3, and 4*. Leeds, Great Britain: E. J. Arnold and Son, Ltd.

Kraitchik, Maurice. *Mathematical Recreations*. New York: Dover Publications, Inc., 1953.

Lieber, Hugh Gray, and Lieber, Lillian R. *The Education of T. C. Mits*. New York: W. W. Norton and Co., Inc., 1944.

McKay, Herbert. *Odd Numbers*. New York: Macmillan Co., 1943.

———. *The World of Numbers*. New York: Macmillan Co., 1946.

Moss, Grace A. *Geometry for Juniors, Books 1, 2, 3, and 4*. Oxford, England: Basil Blackwell and Mott, Ltd., 1960.

Pedoe, Dan. *The Gentle Art of Mathematics*. New York: Macmillan Co., 1959.

Perelman, Yakov. *Figures for Fun*. Distributed by Four Continental Book Corp., Dept. 30, 156 Fifth Ave., New York 10. Moscow U.S.S.R.: Foreign Languages Publishing House, 1957.

Ravielli, Anthony. *An Adventure in Geometry*. New York: Viking Press, 1957.

Reichmann, W. J. *The Fascination of Numbers*. Fair Lawn, New Jersey: Essential Books, Inc. 1957.

Reid, Constance. *From Zero to Infinity*. New York: Thomas Y. Crowell Co., 1960.

School Mathematics Study Group. *Mathematics for Junior High School, Vols. I and II*. New Haven, Connecticut: Yale University Press, 1961.

Spitzer, Herbert F. *Practical Classroom Procedures for Enriching Arithmetic*. St. Louis: Webster Publishing Co., 1956.

Steinhaus, H. *Mathematical Snapshots*. New York: Oxford University Press, Inc., 1960.

Directory of Publishers

The Directory includes the names of the publishers of the books listed in the bibliographies on pages 274-277, followed by the names and addresses of their Canadian representatives.

Allyn and Bacon, Inc., Boston. The Macmillan Co. of Canada Ltd., 70 Bond St., Toronto 2.

Arnold, E. J., and Son Ltd., Leeds, Gt. Britain. The House of Grant (Canada) Ltd., 29 Mobile Dr., Toronto 16.

Blackie and Son Ltd., London. The Ryerson Press, 299 Queen St. W., Toronto 2B.

Blackwell, Basil, and Mott Ltd., Oxford, England. The Copp Clark Publishing Co. Ltd., 517 Wellington St. W., Toronto 2B.

Cassell and Co. Ltd., London. Cassell and Co. (Canada) Ltd., 1068 Broadview Ave., Toronto 6.

Chelsea Publishing Co., New York. Renouf Publishing Co. Ltd., 2182 St. Catherine St. W., Montreal 25, P.Q.

Crowell, Thomas Y., Co., New York. Ambassador Books Ltd., 370 Alliance Ave., Toronto 9.

Day, John, Co., Inc., New York. Longmans Canada Ltd., 20 Cranfield Rd., Toronto 16.

Dial Press, Inc., New York. S. J. Reginald Saunders and Co. Ltd., 266 King St. W., Toronto 2B.

Doubleday and Co., Inc., New York. Doubleday Publishers, 105 Bond St., Toronto 2.

Dover Publications, Inc., New York. McClelland and Stewart Ltd., 25 Hollinger Rd., Toronto 16.

Golden Press, Inc., New York. Musson Book Co. Ltd., 103 Vanderhoof Ave., Toronto 17.

Harvard University Press, Cambridge, Mass. S. J. Reginald Saunders and Co. Ltd., 266 King St. W., Toronto 2B.

Holt, Rinehart and Winston, Inc., New York. Holt, Rinehart and Winston of Canada Ltd., 833 Oxford St., Toronto 18.

Houghton Mifflin Co., Boston. Thomas Nelson and Sons (Canada) Ltd., 91 Wellington St. W., Toronto 1.

Little, Brown and Co., Boston. Little, Brown and Co. (Canada) Ltd., 25 Hollinger Rd., Toronto 16.

Macmillan Company, New York. Collier-Macmillan Canada Ltd., 132 Water St. S., Galt, Ontario.

McGraw-Hill Book Company, New York. McGraw-Hill Co. of Canada Ltd., 253 Spadina Rd., Toronto 4.

Norton, W. W. and Co., Inc., New York. George J. McLeod Ltd., 73 Bathurst St., Toronto 2B.

Oxford University Press, New York. Oxford University Press, 70 Wynford Dr., Don Mills, Ont.

Prentice-Hall, Inc., Englewood Cliffs, New Jersey. Prentice-Hall of Canada Ltd., 520 Ellesmere Road, Scarborough, Ont.

Princeton University Press, Princeton, New Jersey. S. J. Reginald Saunders and Co. Ltd., 266 King St. W., Toronto 2B.

Random House, Inc., New York. Random House of Canada Ltd., 370 Alliance Ave., Toronto 9.

Rinehart and Company, New York. Holt, Rinehart and Winston of Canada Ltd., 833 Oxford St., Toronto 18.

Scott, Foresman and Co., Chicago. W. J. Gage Ltd., 1500 Birchmount Rd., Scarborough 4, Ont.

Simon and Shuster, Inc., New York. Musson Book Co., Ltd., 103 Vanderhoof Ave., Toronto 17.

University of Chicago Press, Chicago. University of Toronto Press, Front Campus, University of Toronto, Toronto 5.

Van Nostrand, D., Co., Inc., Princeton, New Jersey. D. Van Nostrand Co. (Canada) Ltd., 25 Hollinger Rd., Toronto 16.

Viking Press, Inc., The, New York. The Macmillan Co. of Canada Ltd., 70 Bond St., Toronto 2.

Wiley, John and Sons, Inc., New York. General Publishing Co. Ltd., 200 Adelaide St. W., Toronto; and Renouf Publishing Co. Ltd., 2182 St. Catherine St. W., Montreal 25, P.Q.

Yale University Press, New Haven, Connecticut. McGill University Press, 3458 Redpath St., Montreal 25, P.Q.

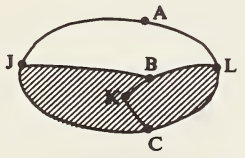
Responses to Special challenges

■ Responses to all "Special challenge" exercises in *STM 1* are given on this page and the next three pages. These responses are not included in the *Response Verification Books* for *STM 1*. Thus, students who have access to the response books have responses to all exercises in the student's books except the "Special challenge" exercises.

118 A Instead of drawing pictures of the farmhouses and wells, you may use dots labeled A, B, and C to represent the farmhouses and dots labeled J, K, and L to represent the wells.

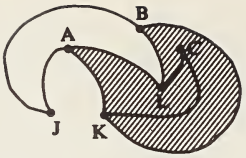
118 B There are many possible sketches.

118 C The sketch below shows one of the possible arrangements of the farmhouses and wells.



118 D No * The sketch above shows how J and L can be connected to each of the farmhouses. The shaded portion shows that K is in the interior of the simple closed curve JBLC and A is in the exterior. Therefore, A and K cannot be connected without crossing the boundary. Another sketch is shown in the next column. Notice that C is in the interior of the simple closed curve BKAL and J is in the exterior.

No matter how you try to connect the wells and farmhouses, you find that you must connect a well and a farmhouse where one is in the interior of a simple closed curve and the other is in the exterior. This means that the farmers cannot arrange the pipes under the conditions given in the problem.



160 A 8

160 B { }, {a}, {b}, {a, b} * 4

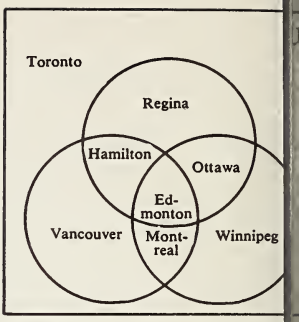
160 C { }, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}, {a, b, c, d} * 16

160 D { }, {a}, {b}, {c}, {d}, {e}, {a, b}, {a, c}, {a, d}, {a, e}, {b, c}, {b, d}, {b, e}, {c, d}, {c, e}, {d, e}, {a, b, c}, {a, b, d}, {a, b, e}, {a, c, d}, {a, c, e}, {a, d, e}, {b, c, d}, {b, c, e}, {b, d, e}, {c, d, e}, {a, b, c, d}, {a, b, c, e}, {a, b, d, e}, {a, c, d, e}, {b, c, d, e}, {a, b, c, d, e} * 32

160 E Notice that a set with two members has 2×2 subsets. A set with three members has $2 \times 2 \times 2$ subsets. A set with four members has $2 \times 2 \times 2 \times 2$ subsets. A set with five members has $2 \times 2 \times 2 \times 2 \times 2$ subsets. Therefore, a set with 6 members has 64 subsets. * A set with 7 members has 128 subsets. * A set with 10 members has 1024 subsets.

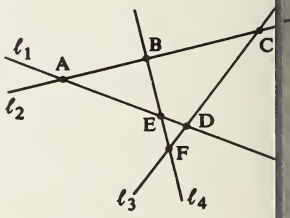
160 F Edmonton—Toronto
Ottawa—Vancouver
Montreal—Regina
Hamilton—Winnipeg

Each sports writer picked a set of teams to win. No team is in the same set as its opponent. Edmonton is in the intersection of the three sets. Therefore, Edmonton did not play any team in the three sets. Since Toronto is the only team not in any of the three sets, Edmonton played Toronto. Ottawa is in the intersection of the first two sets; so Ottawa did not play any team in either of these two sets. Since Edmonton and Toronto played each other, the only team left that Ottawa could play is Vancouver. You can pair the remaining teams in the same way. A Venn diagram like the one at the top of the next column may be used to help find

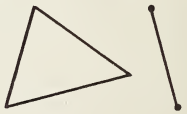


the answer to this problem. The u is the set of eight teams.

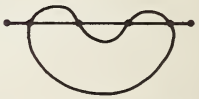
160 G Below is one possible sketch.



189 A The drawing for each of exer through I is one example of a sketch for the exercise.



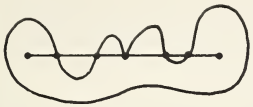
189 B



189 C This is impossible. The inters of a line and a circle can contain, most, only two points.

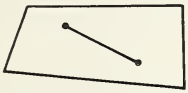
189 D

189 E

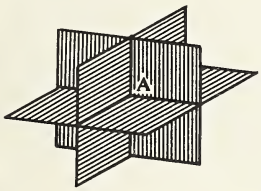


189 F This is impossible. The intersection of two planes is either the empty set or a line.

189 G The segment is included in the plane.



189 H The intersection of the three planes represented below is {A}.



189 I

218 A Mr. Ames lost $44. Use x for the number of dollars the more expensive car cost, y for the number of dollars the less expensive car cost, n for the number of dollars both cars cost, and m for the number of dollars lost or gained on the sale.

A compound condition for the problem is expressed below.
$x + y = n \wedge 80/100 \sim 528/x \wedge 120/100 \sim 528/y$. $\{x | 80/100 \sim 528/x\} = \{660\}$. $\{y | 120/100 \sim 528/y\} = \{440\}$. If x is replaced by 660 and y by 440 in $x + y = n$, then $660 + 440 = n$. $\{n | 660 + 440 = n\} = \{1100\}$. Now the student knows that the total cost was $1100. He also knows that Mr. Ames sold each car for $528, or both cars for $1056. Since $1056 is less than the total cost of the cars, he must find $\{m | 1100 - 1056 = m\}$. The solution set of $1100 - 1056 = m$ is $\{44\}$.

218 B 5 pounds, 9 pounds, and 10 pounds. Name the rocks A, B, and C. A is the lightest rock, and C is the heaviest rock. A and B weigh 14 pounds. A and C weigh 15 pounds. Therefore, C weighs one pound more than B. Since B and C weigh 19 pounds, C weighs 10 pounds and B weighs 9 pounds. Since A and B weigh 14 pounds, A weighs 5 pounds.

237 You should see at once that the numerals are written using a base other than ten. Since only the digits 0, 1, 2, 3, 4, and 5 appear in the numerals, it is quite probable that the base is six. Converting the numerals from base six to base ten, the article reads as follows:

"Babe Ruth was one of the greatest players in the history of baseball. He was born February 6, 1895, and died August 16, 1948, at the age of 53. He played on a major league baseball team from 1914 through 1938. He played with the New York Yankees from 1920 through 1934. Babe Ruth set many records during his lifetime of baseball. Some of these records are: 1330 strike-outs, 2056 bases on balls, 2209 runs batted in, and 1356 extra-base hits.

"Perhaps his most impressive record was set in 1927. In that year, Babe Ruth set a record for the number of home runs hit during one season in the major leagues. He hit 60 home runs in 154 games. This record has stood for 34 years!"

241 A Observe that the numeral "do" is the name of the number 0. Re, mi, fa, sol, la, ti, re-do, re-re, re-mi, re-fa, re-sol, re-la, re-ti, mi-do, mi-re, mi-mi, mi-fa, mi-sol, mi-la, mi-ti, fa-do, fa-re, fa-mi, fa-fa, fa-sol

241 B ti-fa, ti-sol, ti-la, ti-ti, re-do-do, re-do-re, re-do-mi, re-do-fa, re-do-sol, re-do-la, re-do-ti, re-re-do, re-re-re, re-re-mi, re-re-fa, re-re-sol

241 C ti-ti-sol, ti-ti-la, ti-ti-ti, re-do-do-do, re-do-do-re, re-do-do-mi, re-do-do-fa, re-do-do-sol, re-do-do-la, re-do-do-ti, re-do-re-do, re-do-re-re, re-do-re-mi, re-do-re-fa, re-do-re-sol

241 D An example is given below. The numeral "no" is the name of the number 0. The names of the numbers from 1 through 27 are box, dog, fox, got, hot, lot, pox, rot, bono, bobox, bodog, bofox, bogot, bohot, bolot, bopox, borot, dono, dobox, dodog, dofox, dogot, dohot, dolot, dopox, dorot, fono. The names of the numbers from 60 through 70 are lolot, lopox, lorot, pono, pofox, podog, pofix, pogot, pohot, polot, popox. The names of the numbers from 510 through 521 are lodolot, lodopox, lodorot, lofono, lofobox, lofodog, lofofox, lofogot, lofohot, lofolot, lofopox, loforot.

252 A $E = \{0, 2, 4, \ldots\}$. To find the members of E onto which you map the members of N, multiply each member of N by 2. To find the members of N onto which you map the members of E, divide each member of E by 2. * The diagram below represents a way in which set E and set N can be put in one-to-one correspondence.

$\{0, 1, 2, 3, 4, \ldots\}$
$\updownarrow \updownarrow \updownarrow \updownarrow \updownarrow$
$\{0, 2, 4, 6, 8, \ldots\}$

252 B $O = \{1, 3, 5, \ldots\}$. To find the members of O onto which you map the members of N, first multiply each member of N by 2 and then add 1 to the product. To find the members of N onto which you map the members of O, first subtract 1 from each member of O and then divide the difference by 2. * The diagram below represents a way in which set O and set N can be put in one-to-one correspondence.

$\{0, 1, 2, 3, 4, \ldots\}$
$\updownarrow \updownarrow \updownarrow \updownarrow \updownarrow$
$\{1, 3, 5, 7, 9, \ldots\}$

252 C Yes * Yes * Each of the sets E and O can be put in one-to-one correspondence with the set of natural numbers.

252 D Yes * Since the set of natural numbers is an infinite set, a set that can be put

in one-to-one correspondence with the set of natural numbers must be infinite.

252 E Set S is the set whose members are second powers of the members of N. Set V is the set of numbers that have 3 as a factor and that are greater than 2 and less than 523. Set C is the set of counting numbers. Set N × N is the Cartesian set of N.

252 F Set T is a denumerable set. The diagram below represents a way in which set T and set N can be put in one-to-one correspondence.

$\{1,\ 10^1,\ 10^2,\ 10^3,\ \ldots\}$
$\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow$
$\{0,\ 1,\ 2,\ 3,\ \ldots\}$

Rule: Map 1 in set T onto 0 in set N. Then map each of the other members of set T onto the member of N that is the exponent of the member of T.

Set S is a denumerable set. The diagram below represents a way in which set S and set N can be put in one-to-one correspondence.

$\{0,\ 1,\ 4,\ 9,\ 16,\ \ldots\}$
$\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow$
$\{0,\ 1,\ 2,\ 3,\ 4,\ \ldots\}$

Rule: To find the members of S onto which you map the members of N, find the second power of each member of N.

Set V is not a denumerable set. Since V is a finite set, it cannot be put in one-to-one correspondence with N.

Set C is a denumerable set. The diagram below represents a way in which set C and set N can be put in one-to-one correspondence.

$\{1,\ 2,\ 3,\ 4,\ \ldots\}$
$\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow$
$\{0,\ 1,\ 2,\ 3,\ \ldots\}$

Rule: To find the members of N onto which you map the members of C, subtract 1 from each member of C.

Set N × N is a denumerable set. The diagram below represents a way in which set N × N and set N can be put in one-to-one correspondence. The rule is best illustrated by the diagram.

$\{(0,0),\ (0,1),\ (1,0),\ (0,2),\ (1,1),$
$\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow$
$\{\ 0,\ 1,\ 2,\ 3,\ 4,$
$(2,0),\ (0,3),\ (1,2),\ (2,1),\ (3,0),$
$\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow$
$5,\ 6,\ 7,\ 8,\ 9,$
$(0,4),\ (1,3),\ \ldots\}$
$\updownarrow\ \updownarrow$
$10,\ 11,\ \ldots\}$

269 A a was replaced by 5 and b by 7.

269 B a was replaced by 9 and b by 4.

269 C a was replaced by 69 and b by 141.

269 D N is closed under the operation of "star" because for each replacement of a and b, $a * b$ is a member of N. You know that $a * b$ is the same as $a + 2b$. $2b$ is a member of N because of the closure property of multiplication of natural numbers. You also know that $a + 2b$ is a member of N because of the closure property of addition of natural numbers.

269 E $3 * 4 = 4 * 3$. * Yes * Yes

269 F No * No. $3 + 2(4) = 3 + 8$, or 11. $4 + 2(3) = 4 + 6$, or 10. Therefore, $3 * 4$ does not equal $4 * 3$.

269 G $7 * 15 = 15 * 7$. * No. $7 * 15 = 7 + 2(15)$, or 37. $15 * 7 = 15 + 2(7)$, or 29. Therefore, $7 * 15$ does not equal $15 * 7$.

269 H No * For each replacement of a and b, a true statement is not always obtained from $a * b = b * a$.

269 I Not commutative * Three examples follow. Replace a by 3 and b by 5. $3 \P 5 = 3(3) + 5$, or 14. $5 \P 3 = 3(5) + 3$, or 18. Therefore, $3 \P 5$ does not equal $5 \P 3$. Replace a by 8 and b by 2. $8 \P 2 = 3(8) + 2$, or 26. $2 \P 8 = 3(2) + 8$, or 14. Therefore, $8 \P 2$ does not equal $2 \P 8$. There are some replacements for which $a \P b = b \P a$. For example: $9 \P 9 = 9 \P 9$. However, to be a property, you know that the condition must be true for every replacement of the variables.

269 J Not commutative * Replace a by 1 and b by 2. $1 \dagger 2 = 4(1) - 2$, or 2. $2 \dagger 1 = 4(2) - 1$, or 7. Therefore, $1 \dagger 2$ is not equal to $2 \dagger 1$. Replace a by 12 and b by 15. $12 \dagger 15 = 4(12) - 15$, or 33. $15 \dagger 12 = 4(15) - 12$, or 48. Therefore, $12 \dagger 15$ does not equal $15 \dagger 12$. Replace a by 4 and b by 20. $4 \dagger 20 = 4(4) - 20$, which is not a natural number. $20 \dagger 4 = 4(20) - 4$, or 76. Therefore, $4 \dagger 20$ does not equal $20 \dagger 4$.

269 K Commutative * Replace a by 4 and b by 5. $4 \bowtie 5 = 2(4) + 2(5)$, or 18. $5 \bowtie 4 = 2(5) + 2(4)$, or 18. Therefore, $4 \bowtie 5 = 5 \bowtie 4$. Replace a by 3 and b by 1. $3 \bowtie 1 = 2(3) + 2(1)$, or 8. $1 \bowtie 3 = 2(1) + 2(3)$, or 8. Therefore, $3 \bowtie 1 = 1 \bowtie 3$. Replace a by 25 and b by 38. $25 \bowtie 38 = 2(25) + 2(38)$, or 126. $38 \bowtie 25 = 2(38) + 2(25)$, or 126. Therefore, $25 \bowtie 38 = 38 \bowtie 25$.

269 L Not commutative * Replace a by 5 and b by 10. $5 \# 10 = 5(5) - 3(10)$, which is not a natural number. $10 \# 5 = 5(10) - 3(5)$, or 35. Therefore, $5 \# 10$ does not equal $10 \# 5$. Replace a by 2 and b by 3. $2 \# 3 = 5(2) - 3(3)$, or 1. $3 \# 2 = 5(3) - 3(2)$, or 9. Therefore, $2 \# 3$ does not equal $3 \# 2$. Replace a by 18 and b by 12. $18 \# 12 = 5(18) - 3(12)$, or 54. $12 \# 18 = 5(12) - 3(18)$, or 6. Therefore, $18 \# 12$ does not equal $12 \# 18$.

293 A

⊕	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

293 B

⊗	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

293 C {2}

293 D {2}

293 E {0, 1, 2, 3, 4}

293 F {1}

293 G {0}

293 H {2}

293 I {0, 1, 2, 3, 4}

293 J {0}

293 K Yes * The universe for x is T. For each replacement of x, $x \oplus 0 = x$.

293 L Yes * The universe for x is T. For each replacement of x, $x \otimes 1 = x$.

293 M 2 293 P 2 293 S 2

293 N 2 293 Q 2 293 T 4

293 O 4 293 R 4 293 U 2

293 V Yes. If a and b are any two members of T, then $a \ominus b$ is always a member of T. * Yes. If a and b are any two members of T, then $a \oplus b$ is always a member of T.

293 A {3} 293 E {4} 293 H {1}

293 B {3} 293 F {3} 293 I {2}

293 C {3} 293 G {3} 293 J {0}

293 D {3}

323 A p_4 represents 7, p_5 represents 11, p_6 represents 13, and p_7 represents 17.

323 B Yes * It is assumed that set P contains all the primes, and it is further assumed that P is a finite set. Since the members of set P are ordered by the idea of "less than," the last member of set P must be the greatest prime.

324 C Yes * Every counting number greater than 1 is divisible by at least one prime. If P contains all the primes, every counting number must be divisible by at least one member of P.

324 D 0

324 E $p_1 \times p_3 \times p_4 \times p_5 \times \ldots \times p_k$, remainder 1 * $p_1 \times p_2 \times p_4 \times p_5 \times \ldots p_k$, remainder 1 * $p_1 \times p_2 \times p_3 \times p_5 \times p_6 \times \ldots \times p_k$, remainder 1 * $p_1 \times p_2 \times p_3 \times p_4 \times p_5 \times \ldots \times p_{k-1}$, remainder 1

324 F When you divide c by any member of P, you always get a remainder of 1. * No * Every counting number is divisible by each of its factors. You know that c is not divisible by any member of P.

324 G c * No * Yes. Since c is the product of the members of P plus 1, c is greater than each member of P.

324 H You can express every composite number as the product of primes.

324 I You know that P contains each prime that is less than or equal to p_k. You also know that no member of P is a factor of c. Therefore, each prime factor of c must be greater than p_k.

324 J If c is prime, then c is a prime greater than p_k. If c is composite, then each of the prime factors of c is greater than p_k.

324 K No. For any finite set of primes, there is always a prime that the set does not contain. * Yes. If the set of primes is not finite, then it must be infinite.

387 A Yes * Examples are 5 and 7, 11 and 13, 17 and 19, 29 and 31, 41 and 43, and 59 and 61.

387 B 4 * Yes * One

387 C No * 1 is not a prime.

387 D No * Each even number that is greater than 2 has at least three factors: 1, 2, and itself. A prime number has only two factors.

387 E 2 and 3 * No * For each pair of successive natural numbers, one of them is even. The only even number that is a prime is 2.

387 F The successor of 1 is 2, and the predecessor of 1 is 0.

387 G The successor of 27 is 28, and the predecessor of 27 is 26.

387 H The successor of 656 is 657, and the predecessor of 656 is 655.

387 I The successor of 1975 is 1976, and the predecessor of 1975 is 1974.

387 J Successor of a multiple of 3

387 K Multiple of 3

387 L Predecessor of a multiple of 3

387 M Predecessor of a multiple of 3

387 N Multiple of 3

387 O Predecessor of a multiple of 3

387 P Predecessor of a multiple of 3

387 Q Successor of a multiple of 3

387 R 15 * 17 * 13

387 S 27 * 23 * 25

387 T 45 * 47 * 43

387 U 63 * 65 * 61

387 V 75 * 77 * 79

387 W 105 * 101 * 103

387 X The universe for k is the set of odd numbers. Then k, $k+2$, and $k+4$ represent any three consecutive odd numbers. Since each odd number is a multiple of 3, a successor of a multiple of 3, or a predecessor of a multiple of 3, the replacement for k must be a multiple of 3, a successor of a multiple of 3, or a predecessor of a multiple of 3. Now consider each of these three possibilities. If k is replaced by a multiple of 3, then the first of the three consecutive odd numbers, k, $k+2$, and $k+4$, is a multiple of 3. If k is replaced by a successor of a multiple of 3, then the second of the three consecutive odd numbers, k, $k+2$, and $k+4$, is a multiple of 3. For example, if k is replaced by 10, then $k+2=12$. If k is replaced by a predecessor of a multiple of 3, then the third of the three consecutive odd numbers, k, $k+2$, and $k+4$, is a multiple of 3. For example, if k is replaced by 17, then $k+4=21$. Since these are the only possibilities, for any three consecutive odd numbers, one of them must be a multiple of 3.

387 Y Each member of a set of primes must be an odd number, and every odd number is a multiple of 3, or a successor of 3, or a predecessor of 3. Because 3 is the only multiple of 3 that is a prime, {3, 5, 7} is the only set of triple primes.

Your notes

Index

The purpose of this index is to help you locate material in this *Teaching Guide* that pertains to the teaching of various mathematical concepts. When a concept is discussed on two or more successive pages, the reference indicates the first page. For location of subject matter in the student's text, see the index to the text.

Abler students, providing for, *see* Individual differences
Abstract problems, 84, 120, 131, 135, 226, 228
Achievement, evaluation of, 34, 54, 85, 102, 232
Addition
  identity element, 175, 205
  of natural numbers, 160
  operation of, 160, 190
  of rational numbers, 190, 211
"And," connective, 90
Angles, 94
  congruent, 95,
  exterior of, 99
  interior of, 99
  linear pair of, 94
  naming, 94
  right, 95
  sides of, 94
  supplement of, 94
  supplementary, 94
  symbol for, 94
  of triangles, 97
  vertices of, 94
    Anwers to "Special challenge" exercises, 256
Applying mathematics, purpose of and teaching procedures for, 33, 34, 83, 129
Arithmetic mean, 222
Associative property, 166, 200
Average, 222
Axes of a graph, 106

Babylonian numeration system, 141
Base
  of a grouping system, 136
  of a numeration system, 136, 137, 145, 148
  of a place-value numeration system, 137
  of a power, 143
Basic fractions, 183, 193
Betweenness
  for natural numbers, 158
  for points, 60, 63
Bibliographies
  for students, 253
  for teachers, 252
Binary
  numeration system, 139
  operations, 86, 160, 162
Boundary
  half-line, 63
  half-plane, 63
  half-space, 63
  region, 65, 99
Business arithmetic, 129, 132, 225

Cartesian sets, 104
  counting numbers, 116
  formation of, 105
  graphs of, 106
  natural numbers, 110
  non-zero rational numbers, 223
  symbol for, 105
Charts, used to find solution sets, 112, 113, 118
Checking up, purpose and administration of, 34, *see also* Tests
Chord of a circle, 65
Circles, 65
  centre of, 65
  chord of, 65
  diameter of, 65
  exterior of, 65
  interior of, 65
  radius of, 65
  standard description of, 65
Closed region, 100
Closed sentences, 42, 46
Closure property, 161, 162, 188, 190
Code numerals, 136
Code numeration system, 136
Collinear points, 59, 61
Commission, 129
Common denominators, 190, 194
Common factors, 193
Commutative property, 163, 199
Comparison
  with one, 226
  of per cents, 129
  represented by per cents, 129
  represented by rate pairs, 127, 129
  of segments, 63, 65
Complete factorization, 179, 193
Components
  ordered pairs, 103
  per cents, 129, 224
  rate pairs, 123, 221
Composite numbers, 177
Compound condition, 90, *see also* Conditions
Computation
  with decimals, 211
  with mixed numerals, 197, 201, 203
  involving natural numbers, 69, 71, 72, 74
  in a non-decimal numeration system, 147
  involving rational numbers, 187, 190, 192, 193, 197, 207, 211, 213
Conditions, 46
  compound, 90, 113, 117, 118, 133, 135, 218, 219, 223, 225
  for equality, 48, 75, 79, 218, 220
  for equivalence, 126, 127, 131, 221, 222, 225, 226, 228
  geometric, 65
  for inequality, 50, 77, 81, 218
  simple, 79, 81, 112, 115, 126, 131, 218, 220, 221, 225, 226
  solution sets of, 48, 90
  in two variables, 103, 112, 113, 115, 117, 118, 133, 135, 223, 225
Congruence, 62
Congruent
  angles, 95
  segments, 62, 65
  triangles, 97
Connectives
  "and," 90
  "or," 91
Contrapositive of a property, 100
Converse of a property, 61
Converting
  decimals to fraction numerals, 213
  fraction numerals to decimals, 213
  fraction numerals to mixed numerals, 197
  mixed numerals to fraction numerals, 197
  numerals from one base to another base, 145
Coplanar points, 65

Correspondence, *see* One-to-one correspondence
Counting numbers, 82
  Cartesian sets of, 116
  common factors, 193
  complete factorization, 179
  composite, 177
  as factors, 143
  greatest common factor of, 180
  prime, 177, 187
  relatively prime, 183
  unique factorization of, 179

Decimal numerals, 143, 149, 209
  computing with, 211
  converting to fraction numerals, 213
  repeating, 214
  repetend in, 214
  terminating, 214
Denominator of a fraction, 182
Density property, 216
Denumerable sets, 156
Diameter, 65
Difference
  of natural numbers, 161
  property, 192
  of rational numbers, 192 211
Digits, 139
Discount, 129
Disjoint sets, 87
Distributive property, 167, 202
Division
  of natural numbers, 71, 72, 163
  partial quotient in, 73
  of rational numbers, 207, 212
  related to repeated subtractions, 72
  symbol for, 175, 208
Duodecimal numeration system, 139

Egyptian numeration system, 141
Empty set, 48, 51
Endpoints of
  chord, 65
  radius, 65
  rays, 94
  segment, 62
Equal segments, 62
Equal sets, 49
Equality, 48, 198
Equation, 48
Equilateral triangles, 97
Equivalence, 122
Equivalent fractions, 182, 183
Equivalent ordered pairs, 121
  finding of, 122
  as members of a proportional relation, 121
  as rate pairs, 121
Equivalent rate pairs, 121, 124, 126, 221, 222, 224
Equivalent sets, 152, 155, 156
Eratosthenes, Sieve of, 177
Evaluation of achievement, 34, 54, 85, 102, 252
Even numbers, 154, 156, 177
Expanded form of a numeral, 143, 145, 209
Exponential notation, 143, 149
Exponents, 143
Expressions, simplifying of, 169
Exterior of
  angle, 99
  circle, 65
  simple closed curve, 100
  simple open curve, 100
  triangle, 99

Factorization, 179
Factors, 143, 179
  common, 193
  greatest common, 180, 193
  prime, 179
False statements, 46
Finite sets, 51
Fraction numerals, 182
  converting to decimals, 213
  converting to mixed numerals, 197
Fractions, 181
  basic, 183, 193
  common denominator, 190, 194
  denominators, 182
  equivalent, 182, 183
  least common denominator, 194
  numerators, 182
  reduction property of, 183
  symbol for, 182
  "three cases" of, 227

Geometric condition, 65
Geometric figure, 55
Geometry
  nonmetric, 55
  in the *Seeing Through Mathematics* program, 14
  undefined terms in, 55, 62
Graphs
  axes of, 106
  of Cartesian sets, 106
  incomplete, 68
  of natural numbers, 67, 106
  of ordered pairs, 106
  of a proportional relation, 223
  of rational numbers of arithmetic, 185
  of solution sets, 68, 91, 106, 113, 218
Greater than
  for natural numbers, 157
  for rational numbers, 186
  for segments, 63
  symbol for, 50
Greatest common divisor, 193
Greatest common factor, 180, 193
Grouping numeration system, 136

Half-line, 63, 94
Half-plane, 63
Half-space, 63

Identity-element property, 175, 205
Identity elements, 175, 205
Image, 151
Individual differences, providing for, 34, 35, 36 (Each lesson note in this *Teaching Guide* contains suggestions for handling individual differences in a section labeled "Provision for individual differences.")
Individual study, 39
Inequality, 50
Infinite proportional relation, 125
Infinite sets, 51
Interest, 129
Interior of
  angle, 99
  circle, 65
  simple closed curve, 100
  simple open curve, 100
  triangle, 99
Intersecting lines, 93, 95
Intersecting planes, 93
Intersection, 86
  of half-planes, 100
  of lines, 93, 95
  of lines and planes, 93
  of planes, 93
  of sets, 86, 88, 90, 93
  of solution sets, 113
  symbol for, 86
Isomorphism, 195, 197, 208
Isosceles triangles, 97

Keeping skillful, purpose of and teaching procedures for 33, 74, 79

Least common denominator, 194
Least common multiple, 194
Less than
  for natural numbers, 157
  for rational numbers, 186
  for segments, 63
  symbol for, 50

Line segments, *see* Segments
Linear pair of angles, 94,
Lines, 56
  half-, 63, 94
  intersecting, 93, 95
  intersection of planes and, 93
  naming, 60
  parallel, 93
  perpendicular, 95
  relating planes and, 58, 93
  relating points and, 58
  separation of, 63
  skew, 93
  symbol for, 59

Maintenance of mathematical ideas and skills, 33, 36, 74, 83
Mapping
  into, 151
  many-to-one, 151, 161, 162, 187, 190
  one-to-one, 151, 195
  onto, 151
Mates, 195, 197
Member of a set, 44
Mixed numerals, 197
Modular arithmetic, 172
Multiple, 194
Multiplication
  bases other than base ten, 147
  identity element, 175, 205
  of natural numbers, 69, 162
  operation of, 162, 187
  by powers of ten, 211
  of rational numbers, 187, 211
  symbol for, 169

Name-referent distinction, 42
Natural-number line, 67
Natural-number system, 170
Natural numbers, 51
  addition of, 160
  associated with sets, 155
  betweenness for, 158
  Cartesian set of, 110
  as components of fractions, 182
  difference of, 161
  division of, 71, 72, 163
  factorization of, 179
  graphs, 67, 106
  "greater than" for, 157
  least common multiple of, 194
  "less than" for, 157
  as mates of rational numbers, 195, 197
  multiples of, 194
  multiplication of, 69, 162
  ordering, 157
  product of, 69, 162
  properties, 157, 161, 162, 163, 166, 167, 175
  quotient of, 71, 72, 163
  set of, 51
  subtration of, 161
  successor of, 158
  sum of, 160
  tabulation of set of, 51
Noncollinear points, 59
Nonmetric geometry, 55
Non-zero rational numbers, 208, 221
Not equal to, 50
Number lines
  natural-number, 67
  rational-number, 185, 216
Number systems
  finite, 171, 172
  natural, 170
  rational, 203
Number theory, 177, 179 187, 215
Numbers, 42
  associated with points, 67, 185
  counting, 82
  expressed by exponential notation, 143, 149
  expressed by scientific notation, 149
  natural, 51
  rational numbers of arithmetic, 183
  rounding, 71, 212
Numerals, 42
  code, 136
  converting, 145, 197, 213
  decimal, 143, 149, 209, 213
  digits in, 139
  expanded form of, 143, 145, 209
  fraction, 182
  mixed, 197
  in scientific notation, 149
  tally, 135
Numeration systems, 135
  Babylonian, 141
  bases of, 136, 137, 145, 148
  binary, 139
  code, 136
  computation in, 147
  duodecimal, 139
  Egyptian, 141
  grouping, 136
  place-value, 137, 144
  properties of, 135, 137, 141
  Roman, 141
  tally, 135
Numerator of a fraction, 182

Odd numbers, 154, 156, 177
Omission of exercises
  (The lesson notes in this *Teaching Guide* contain suggestions for omissions.)
One-to-one correspondence, 151
  between two finite sets, 151
  between two infinite sets, 154, 156, 195
Open region, 100
Open sentences, 42, 46
Operation, 160
  of addition, 160, 190
  binary, 86, 160, 162
  of multiplication, 162, 187
"Or," connective, 91
Ordered pairs, 103
  associated with points, 106
  components of, 103
  equivalent, 121
  finding of equivalent, 122
  as fractions, 182
  graphs of, 106
  as members of a proportional relation, 121
  as rate pairs, 121
  sets of, 104, 106, 121, 223
  symbol for, 105
Order property, 157, 186

Parallel lines, 93
Parallel planes, 93
Parentheses, use of, 169
Per cents, 129
  comparison of, 129
  components of, 129, 224
  of decrease, 225
  of increase, 225
  less than 1%, 224
  related to rate pairs with second component of one, 228
  symbol for, 129
  "three cases" of, 131
Perpendicular lines, 95
Placeholder, 42, 46
Place-value, property of, 137
Place-value numeration system, 137, 144
Planes, 56
  half-, 63
  intersection of, 93
  intersection of lines and, 93
  naming, 60
  parallel, 93
  regions of, 65, 99
  relating lines and, 58, 93
  relating points and, 58
  separation of, 63, 65, 99
Points, 55
  associated with numbers, 67, 185
  associated with ordered pairs, 106
  betweenness for, 60, 63

collinear, 59, 61
coplanar, 65
noncollinear, 59
relating lines and, 58
relating planes and, 58
sets of, 55, 60, 63, 65, 93, 94, 97, 99
Polygon, *see* Simple polygon
Power, 143
Power set, 119
Prime numbers, 177, 187
Principal, 132
Problem solving, 16, 33
abstract problems, 84, 120, 131, 135, 226, 228
arithmetic mean, 222
business arithmetic, 129, 132, 225
compound conditions, 92, 117, 118, 127, 133, 135, 223, 225
conditions for equality, 74, 79, 84, 116, 117, 220
conditions for equivalence, 127, 131, 133, 222, 225, 226, 228
conditions for inequality, 77, 81, 115
"fraction of" expressions, 226
per cents, 131, 225, 228
Product
of natural numbers, 69, 162
of rational numbers, 187, 211
Proofs, in the *Seeing Through Mathematics* program, 16
Proper subset of a set, 157
Properties
associative, 166, 200
betweenness for points, 61
closure, 161, 162, 188, 190
commutative, 163, 199
congruence, 62
contrapositive of, 100
converse of, 61
density, 216
difference, 192
distributive, 167, 202
"greater than" for segments, 64
grouping system, 136
identity-element, 175, 205
intersecting lines, 93
intersecting planes, 93
"less than" for segments, 64
natural numbers, 157, 161, 162, 163, 166, 167, 175
numeration systems, 135, 137, 141
ordering numbers, 157, 186
parallel lines, 93
parallel planes, 93
place-value, 137
quotient, 208
rational numbers of arithmetic, 188, 190 192, 199, 200, 202, 205, 206, 208, 216
reciprocal, 206
reduction, 183
segments, 64
simple closed curve, 100
skew lines, 93
successor, 158
sum, 190
tally system, 135
that relate lines and planes, 58, 93
that relate points and lines, 58
that relate points and planes, 58
transitive, 62, 64, 198
unique factorization, 179
well-defined, 188, 190
zero, 175, 205
Proportional relations, 121
graphs of, 223
infinite, 125
members of, 121
standard description, 125
tabulation of, 125

Quotient
expressed by a decimal, 212
of natural numbers, 71, 72, 163
property, 208
of rational numbers, 207 212
rounding, 212
symbol for, 208

Radius, 65
Rate
of commission, 132
of discount, 132
of interest, 132
represented by rate pairs, 127
Rate pairs, 121
components of, 123, 221
equivalent, 121, 124, 126, 221, 222, 224
finding equivalent, 122, 126, 221, 222, 224
with second component of one, 222, 226, 228
sets of, 121, 223
symbol for, 125
Ratio, 122, 125, 129
Rational-number line, 185, 216
Rational-number system, 203
Rational numbers of arithmetic, 183
addition of, 190, 211
as components of per cents, 224
as components of rate pairs, 221, 222
density property of, 216
difference of, 192, 211
division of, 207, 212
equal, 183
expressed by decimals, 209, 211
expressed by mixed numerals, 197
graphs, 185
"greater than" for, 186
"less than" for, 186
as mates of natural numbers, 195, 197
multiplication of, 187, 211
non-zero, 208, 221
ordering, 186
product of, 187, 211
properties, 188, 190, 192, 199, 200, 202, 205, 206, 208, 216
quotient of, 207, 212
reciprocals of, 206, 212
subtraction of, 192, 211
sum, of 190, 211
Rays, 94
endpoint of, 94
naming, 94
symbol for, 94
union of, 94
Reciprocal property, 206
Reciprocals, 206, 212
Reduction property, 183
Referent, 42
Regions
closed, 100
open, 100
of a plane, 65, 99
Relatively prime numbers, 183
Repeating decimals, 214
Repetend, 214
Repetition, property of, 135
Right angle, 95
Right triangle, 97
Roman numeration system, 141
Rounding numbers, 71, 212

Scalene triangle, 97
Scientific notation, 149
Seeing Through Mathematics, Book 1
overview of, 17
special features of, 30
Seeing through Mathematics program, overview of, 12
Segments, 62
comparison of, 63, 65
congruent, 62, 65
endpoints of, 62
equal, 62
"greater than" for, 63
"less than" for, 63
symbol for, 62
union of, 97, 100

Sentences, 42, 46
Separation, 63
of a line, 63
of a plane, 63, 65, 99
of space, 63
Service charge, 129
Sets, 44
Cartesian, 104, 106, 110, 223
common property of, 155
contained in, 56
denumerable, 156
disjoint, 87
empty, 48, 51
equal, 49
equivalent, 152, 155, 156
of equivalent fractions, 182, 183
of equivalent ordered pairs, 121
finite, 51
included in, 56
infinite, 51
intersection of, 86, 88, 90, 93
isomorphic, 195
member of, 44
natural numbers associated with, 155
one-to-one correspondence between two, 151, 156, 195
or ordered pairs, 104, 106, 121, 223
of points, 55, 60, 63, 65, 93, 94, 97, 99
power set of, 119
proper subset of, 157
of rate pairs, 121, 223
solution, 48, 90
standard, 155
subset of, 45, 157
symbol for, 44
tabulation of, 45, 51
union of, 88
Sieve of Eratosthenes, 177
Simple closed curve, 99
Simple condition, *see* Conditions
Simple open curve, 99
Simple polygon, 100
Skew lines, 93
Slow learners, providing for, *see* Individual differences
Solution of a condition, 52
Solution sets of conditions, 48, 90
graphs of, 68, 91, 106, 113, 218
intersection of, 113
sets of points as, 65
standard description of, 53, 218
use of charts in finding, 112, 113, 118
Space, 56
half-, 63
separation of, 63
Special challenge
purpose of and teaching procedures for, 31, 101
responses to, 256
Standard description
of a circle, 65
of a proportional relation, 125
of a solution set, 53, 110, 218
symbol for, 53
Standard sets, 155
Statements, 46
Subsets, 45, 157
Subtraction
of natural numbers, 161
of rational numbers, 192, 211
Successor of a natural number, 158
Successor property for natural numbers, 158
Sum
of natural numbers, 160
property, 190
of rational numbers, 190, 211
Supplement, angle, 94
Supplementary angles, 94,
Supplementary tests, 252
Symbols
and, 91
angle, 94
Cartesian set, 105
congruence, 62
division, 175, 208
empty set, 48
equality, 48
equivalence, 125
fraction, 182
greater than, 50
intersection, 86
less than, 50
line, 59
multiplication, 169
not equal to, 50
ordered pair, 105
per cent, 129
quotient, 208
rate pair, 125
ray, 94
repetend, 214
segment, 62
set, 44
standard description, 53
triangle, 97
union, 88

Tabulation of sets, 45, 51
Take-home pay, 129
Tally numerals, 135
Talley system of numeration, 135
Teaching procedures, 34, 37 (Each lesson note in this *Teaching Guide* contains suggestions for teaching procedures in the sections labeled "Instruction notes" and "Provision for individual differences.")
Terminating decimals, 214
Tests, purpose and administration of
cumulative, 34, 102
end-of-block, 34, 54
end-of-unit, 34, 85
inventory, 69
supplementary, 232
Transitive property, 62, 64, 198
Triangles, 97
angles of, 97
congruent, 97
equilateral, 97
exterior of, 99
interior of, 99
isosceles, 97
naming, 97
right, 97
scalene, 97
side of, 97
symbol for, 97
vertices of, 97
Triple primes, 177, 215
True statements, 46
Twin primes, 177

Union, 88
of segments, 97, 100
of sets, 88
symbol for, 88
of two rays, 94
Unique factorization, 179
Universe, 48, 65, 103

Variable, 42, 46
Venn diagrams, 86, 88, 91
Vertex of an angle, 94
Vertices of a triangle, 97

Well-defined property, 188, 190

Zero property, 175, 205

Date Due			
RETURNED FEB 25 '70		APR 9	RETURN
FEB 23 '70		NOV 9	RETURN
MAR 5 '70, RETURNED			
MAR 5 '70			
JUN 12 '70 RETURNED			
MAY 4 '70			
CURR OC 28 '70			
CURR NO 3 '70			
CURR DE 3 '70			
[redacted] AL			
DEC 8 '70			
CURR DE 15 '70 RENEWAL DEC 15 '70			
CURR AP 30 '71,			
RETD. APR 17 1971			
FEB 29 RETURN			
MAY 1 RETURN			
MAR 4 RETURN			

QA 11 S46 BK-1 TCH-GD-
SEEING THROUGH MATHEMATICS/
/SPECIAL ED --

NL 39855477 CURR HIST

000019482140

EDUCATION
CURRICULUM

444220

QA
QA 11 S46 bk.1 tch.gd.
Seeing through mathematics /

39855477 CURR

TEACH
MAN

CURRICULUM
EDUCATION LIBRARY

W. J. Gage Limited